On the Approximation of Linear AE-Solution Sets

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Abstract

When considering systems of equations, it happens often that parameters are known with some uncertainties. This leads to continua of solutions that are usually approximated using the interval theory. A wider set of useful situations can be modeled if one allows furthermore different quantifications of the parameters in their domains. In particular, quantified solution sets where universal quantifiers are constrained to precede existential quantifiers are called AE-solution sets.

A state of the art on the approximation of linear AE-solution sets in the framework of generalized intervals (intervals whose bounds are not constrained to be ordered increasingly) is presented in a new and unifying way. Then two new generalized interval operators dedicated to the approximation of quantified linear interval systems are proposed and investigated.

1 Introduction

One of the most fundamental application of interval analysis is to allow one dealing with uncertain parameters (see [12, 15, 8] for some introductions to interval analysis). While a problem may have a discrete number of solutions given a value of its parameters, it is likely to have a manifold of solutions when parameters are constrained to belong to some intervals. When the problem to be solved consists of finding the solutions of systems of equations, considering interval domains for parameters gives rise to the definition of *united solution sets*:

$$\Sigma(f, \mathbf{a}, \mathbf{b}) := \{x \in \mathbb{R}^n | \exists a \in \mathbf{a} \, \exists b \in \mathbf{b} \; f(a, x) = b\}, \quad (1)$$

where vectorial notations are used to obtain a compact expression (i.e. $f : \mathbb{R}^p \cdot \mathbb{R}^n \to \mathbb{R}^m$). In the special case where f is linear, the parameters $(a_i)_{i\in\{1,\ldots,n\}}$ are arranged into a matrix $A := (A_{i,j})_{i,j\in\{1,\ldots,n\}}$ while an interval matrix $\mathbf{A} := (\mathbf{A}_{i,j})_{i,j\in\{1,\ldots,n\}}$ is used to provide the interval domains, leading to the notation

$$\Sigma(\mathbf{A}, \mathbf{b}) := \{x \in \mathbb{R}^n | \exists A \in \mathbf{A} \, \exists b \in \mathbf{b} \; Ax = b\}. \quad (2)$$

An important part of interval analysis is dedicated to the approximation of united solution sets, and in particular of linear united solution sets. Methods like the interval Gauss-Seidel and the Krawczyk operators, the interval Gauss elimination (cf. [15] and references therein), the Hansen-Bliek algorithm (cf. [16] and references therein), etc., are widely used to build some outer approximations of linear united solution sets.

A wider set of useful situations can be modeled by allowing different quantifiers for parameters. A general quantification of parameters is only little studied today in the framework of numerical analysis[1]. Constraining universal quantifiers to precede existential ones offers a compromise between the modeling power and the difficulty of the problems met in the study of the solution set. Such solution sets are called *AE-solution sets*[2] (cf. [28, 3, 5] the approximation of linear and non-linear AE-solution sets and [9] for some applications). While interval analysis is well fitted to the approximation of united solution sets, it has been demonstrated that the formalism of generalized intervals (i.e. interval whose bounds are not constrained to be ordered) is the right framework for the approximation of AE-solution sets.

A state of the art of the methods dedicated to the approximation of linear AE-solution sets using generalized intervals is first proposed. This state of the art introduces a new convention for the representation of quantifiers using generalized intervals, allowing to homogenize the approximation of united solution sets in the framework of classical interval analysis and the study of AE-solution sets in the framework

*This work was mainly done during the author's thesis at the University of Sophia Antipolis and his post-doctoral study at the University of Central Arkansas.

[1] The quantifier elimination, with its well-known limitations, is a formal method which allows studying general quantifications (cf. [2]).

[2] This denomination coming from the succession of universal (All) quantifiers preceded by existential (Exists) ones.

of generalized intervals. Then two new generalized interval operators dedicated to the outer approximation of linear AE-solution sets are presented.

2 Generalized intervals

Generalized intervals are intervals whose bounds are not constrained to be ordered, for example $[-1, 1]$ and $[1, -1]$ are generalized intervals. They have been introduced in [17, 10] (see also [11, 28]) so as to improve the algebraic and the order structures of intervals. The set of generalized intervals is denoted by $\mathbb{KR}$ and is divided into three subsets:

- The set of *proper intervals* with bounds ordered increasingly. These proper intervals are identified with classical intervals. The set of proper intervals is denoted $\mathbb{IR} := \{[a, b] \mid a \leq b\}$. *Strictly* proper intervals satisfy $a < b$.
- The set of *improper intervals* with bounds ordered decreasingly. It is denoted by $\overline{\mathbb{IR}} := \{[a, b] \mid a \geq b\}$. *Strictly* improper intervals satisfy $a > b$.
- The set of *degenerated intervals* $\{[a, b] \mid a = b\} = \mathbb{IR} \cap \overline{\mathbb{IR}}$. Degenerated intervals are identified to reals.

Therefore, from a set of reals $\{x \in \mathbb{R} | a \leq x \leq b\}$, one can build the two generalized intervals $[a, b]$ and $[b, a]$. It will be convenient to switch from one to the other keeping the underlying set of reals unchanged. To this purpose, the following three operations are introduced: the *dualization* is defined by $\operatorname{dual}[a, b] = [b, a]$; the *proper projection* is defined by $\operatorname{pro}[a, b] = [\min\{a, b\}, \max\{a, b\}]$; the *improper projection* is defined by $\operatorname{imp}[a, b] = [\max\{a, b\}, \min\{a, b\}]$.

The generalized intervals are partially ordered by an inclusion which extends the inclusion of classical intervals. Given two generalized intervals $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $\mathbf{y} = [\underline{\mathbf{y}}, \overline{\mathbf{y}}]$, the inclusion is defined by $\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \leq \underline{\mathbf{x}} \wedge \overline{\mathbf{x}} \leq \overline{\mathbf{y}}$. For example, $[-1, 1] \subseteq [-1.1, 1.1]$ (this matches the set inclusion), $[1.1, -1.1] \subseteq [1, -1]$ (the inclusion between the underlying sets of real is reversed for improper intervals) and $[2, 0.9] \subseteq [-1, 1]$. The latter case is known to provide the following interpretation: given $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$, $\mathbf{x} \cap \mathbf{y} = \emptyset$ is equivalent to $\operatorname{dual}\mathbf{x} \subseteq \mathbf{y}$ (which is also equivalent to $\operatorname{dual}\mathbf{y} \subseteq \mathbf{x}$). As degenerated intervals are identified to reals, if $\mathbf{x}$ is proper then $x \in \mathbf{x} \iff x \subseteq \mathbf{x}$. On the other hand, if $\mathbf{x}$ is not proper then for all $x \in \mathbb{R}$ the inclusion $x \subseteq \mathbf{x}$ is false (hence a characterization of a set E of the form $x \in E \Longrightarrow x \subseteq \mathbf{x}$ means that $E \subseteq \mathbf{x}$ in the case where $\mathbf{x}$ is proper, while $E = \emptyset$ in the case where $\mathbf{x}$ is not proper). From this inclusion are defined generalized interval meet and join operation: $[a, b] \wedge [c, d] := [\max\{a, c\}, \min\{b, d\}]$ and $\vee := [\min\{a, c\}, \max\{b, d\}]$. Note that it is important to use different symbols to differentiate these latter operations with the union and intersection of classical intervals: e.g. $[-1, 1] \cap [2, 3] = \emptyset$ while $[-1, 1] \wedge [2, 3] = [2, 1]$.

The generalized interval arithmetic (also called Kaucher arithmetic) extends the classical interval arithmetic. Its definition can be found e.g. in [11, 28]. When only proper intervals are involved, this arithmetic coincides with the interval arithmetic: for $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$ one has $\mathbf{x} \circ \mathbf{y} = \{x \circ y \in \mathbb{R} \mid x \in \mathbf{x}, y \in \mathbf{y}\}$. When proper and improper intervals are involved, some new operations are introduced. For example, $[a, b] + [c, d] = [a + c, b + d]$ and, if $a, b, c, d \geq 0$ then $[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$. Also, $-[a, b] = (-1) \cdot [a, b] = [-b, -a]$.

The generalized interval arithmetic has better algebraic properties than the classical interval arithmetic: the addition in $\mathbb{KR}$ is a group. The opposite of an interval $\mathbf{x}$ is $-\operatorname{dual}\mathbf{x}$, i.e.,

$$\mathbf{x} + (-\operatorname{dual}\mathbf{x}) = \mathbf{x} - \operatorname{dual}\mathbf{x} = [0, 0]. \tag{3}$$

The multiplication in $\mathbb{KR}$ restricted to generalized intervals whose proper projection does not contain 0 is also a group. The inverse of such a generalized interval $\mathbf{x}$ is $1/(\operatorname{dual}\mathbf{x})$, i.e.,

$$\mathbf{x} \cdot (1/\operatorname{dual}\mathbf{x}) = \mathbf{x}/(\operatorname{dual}\mathbf{x}) = [1, 1]. \tag{4}$$

Although addition and multiplication in $\mathbb{KR}$ are associative, they are not distributive. The addition and multiplication in $\mathbb{KR}$ are linked by the following distributivity law, called the *conditional distributivity* (see [29, 19, 28]). Whatever are $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{KR}$,

$$\mathbf{x}\cdot\mathbf{y} + (\operatorname{imp}\mathbf{x})\cdot\mathbf{z} \subseteq \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) \subseteq \mathbf{x}\cdot\mathbf{y} + (\operatorname{pro}\mathbf{x})\cdot\mathbf{z}. \tag{5}$$

The three following particular cases will be of practical interest in this paper:

- *Subdistributivity*: if $\mathbf{x} \in \mathbb{IR}$ then $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
- *Superdistributivity*: if $\mathbf{x} \in \overline{\mathbb{IR}}$ then $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \supseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
- *Distributivity*: if $x \in \mathbb{R}$ then $x \cdot (\mathbf{y} + \mathbf{z}) = x \cdot \mathbf{y} + x \cdot \mathbf{z}$.

Another useful property of the Kaucher arithmetic is its monotonicity with respect to the inclusion: whatever are $\circ \in \{+, \cdot, -, \div\}$ and $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{KR}$,

$$\mathbf{x} \subseteq \mathbf{x}' \wedge \mathbf{y} \subseteq \mathbf{y}' \implies (\mathbf{x} \circ \mathbf{y}) \subseteq (\mathbf{x}' \circ \mathbf{y}'). \tag{6}$$

More specifically the following equivalence holds:

$$\mathbf{x} \subseteq \mathbf{x}' \iff \mathbf{x} + \mathbf{y} \subseteq \mathbf{x}' + \mathbf{y}. \tag{7}$$

Finally, generalized interval vectors $\mathbf{x} \in \mathbb{KR}^n$ and generalized interval matrices $\mathbf{A} \in \mathbb{KR}^{n\times n}$ together with their additions and multiplications are defined similarly to their real and classical interval counterparts. The operations pro and dual are applied componentwise to these objects.

3 Linear AE-solution sets

AE-solution sets generalize united solution sets allowing different quantification of the parameters in their interval domains, with the constraint that universal quantifiers precede existential quantifiers. For example, given $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and $\mathbf{b} \in \mathbb{IR}^n$, the following solution set is an AE-solution set:

$$\begin{aligned}\{x \in \mathbb{R}^2 | \forall A_{11} \in \mathbf{A}_{11}, \forall A_{21} \in \mathbf{A}_{21}, \forall A_{22} \in \mathbf{A}_{22}, \\ \forall b_1 \in \mathbf{b}_1, \exists A_{12} \in \mathbf{A}_{12}, \exists b_2 \in \mathbf{b}_2, Ax = b\}.\end{aligned} \tag{8}$$

As quantifiers of the same kind commute, an AE-solution set is completely defined by one interval and one quantifier for each parameter A_{ij} and b_k. A useful representation of linear AE-solution sets is given in [28]: $\mathbf{A}^\alpha$ and $\mathbf{A}^\alpha$ are defined for $\alpha \in \{\forall, \exists\}$ as

$$(\mathbf{A}^\alpha)_{ij} = \begin{cases} \mathbf{A}_{ij} \text{ if } Q_{ij} = \alpha \\ 0 \text{ otherwise} \end{cases} \tag{9}$$

and

$$(\mathbf{b}^\alpha)_k = \begin{cases} \mathbf{b}_k \text{ if } Q_k = \alpha \\ 0 \text{ otherwise} \end{cases}, \tag{10}$$

where $Q_{ij}, Q_k \in \{\exists, \forall\}$ are the quantifiers of the different parameters. Thanks to these definitions, the general expression of a linear AE-solution can be given in the following way:

$$\begin{aligned}\{x \in \mathbb{R}^n | \forall A^\forall \in \mathbf{A}^\forall, \forall b^\forall \in \mathbf{b}^\forall, \exists A^\exists \in \mathbf{A}^\exists, \\ \exists b^\exists \in \mathbf{b}^\exists, (A^\forall + A^\exists)x = (b^\forall + b^\exists)\}.\end{aligned} \tag{11}$$

For example, the AE-solution set (8) can be defined by

$$\mathbf{A}^\exists = \begin{pmatrix} 0 & \mathbf{A}_{12} \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{A}^\forall = \begin{pmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \tag{12}$$

$$\text{and} \quad \mathbf{b}^\exists = \begin{pmatrix} 0 & \mathbf{b}_2 \end{pmatrix}^{\mathrm{T}} \text{ and } \mathbf{b}^\forall = \begin{pmatrix} \mathbf{b}_1 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{13}$$

Applied with (12) and (13) the definition (11) is obviously equivalent to (8). Finally, a second useful representation of AE-solution sets is introduced using generalized intervals. To each parameter has to be associated both an interval and a quantifier. Generalized intervals provides a very good representation of this couple of data. One associates a generalized interval to each parameter with the following interpretation:

- The proper projection of the generalized interval is the interval domain of this parameter.
- The quantifier associated to the parameter depends on the proper/improper quality of the generalized interval: $\mathbf{A}_{ij} \in \mathbb{IR} \iff Q_{ij} = \exists$ and hence $\inf \mathbf{A}_{ij} > \sup \mathbf{A}_{ij} \iff Q_{ij} = \forall$. Also, $\mathbf{b}_k \in \mathbb{IR} \iff Q_k = \exists$ and hence $\inf \mathbf{b}_k > \sup \mathbf{b}_k \iff Q_k = \forall$.

These two different definitions of linear AE-solution sets are obviously related by

$$\mathbf{A} = \mathbf{A}^\exists + \operatorname{dual} \mathbf{A}^\forall \tag{14}$$

$$\mathbf{b} = \mathbf{b}^\exists + \operatorname{dual} \mathbf{b}^\forall. \tag{15}$$

The convention chosen here to relate quantifiers and proper/improper qualities is different to the one used in [28]. The new convention has the great advantage of unifying the framework of AE-solution sets with the classical framework of united solution sets. The notation $\Sigma(\mathbf{A}, \mathbf{b})$ can now be generalized to AE-solution sets: $\Sigma(\mathbf{A}, \mathbf{b})$ is now defined for $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and $\mathbf{b} \in \mathbb{KR}^n$ and corresponds to the AE-solution set obtained using the conventions stated above. In particular, if $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and $\mathbf{b} \in \mathbb{IR}^n$ then

$$\Sigma(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^m | \exists A \in \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\},$$

so the united solution set is defined as a particular AE-solution set in a homogeneous way. If $\mathbf{A} \in \overline{\mathbb{IR}}^{n\times n}$ and $\mathbf{b} \in \mathbb{IR}^n$ then

$$\Sigma(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^m | \forall A \in \operatorname{pro} \mathbf{A}, \exists b \in \mathbf{b}, Ax = b\}$$

and it is called a tolerable solution set. If $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and $\mathbf{b} \in \overline{\mathbb{IR}}^n$ then

$$\Sigma(\mathbf{A}, \operatorname{dual} \mathbf{b}) = \{x \in \mathbb{R}^m | \forall b \in \mathbf{b}, \exists A \in \mathbf{A}, Ax = b\}$$

and it is called a controllable solution set. As another example, the AE-solution set (8) can be written $\Sigma(\mathbf{A}', \mathbf{b}')$ with

$$\mathbf{A}' = \begin{pmatrix} \operatorname{dual} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \operatorname{dual} \mathbf{A}_{21} & \operatorname{dual} \mathbf{A}_{22} \end{pmatrix} \text{ and } \mathbf{b}' = \begin{pmatrix} \operatorname{dual} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Remark. With these definitions, the new convention for the relationship between quantifiers and proper/improper qualities and the convention proposed in [28] are related by $\Sigma(\mathbf{A}, \mathbf{b}) = \Xi(\operatorname{dual} \mathbf{A}, \mathbf{b})$. This change of notation allows an homogenization between the classical interval analysis and the framework of generalized interval dedicated to the approximation of AE-solution sets. As shown in [4], this homogenization is effective also in the context of non-linear united and AE-solution sets, where a generalized interval Newton operator dedicated to the approximation of non-linear AE-solution sets has been proposed.

Not only do generalized intervals provide us with a very convenient modeling language for AE-solution sets, but they are also a powerful analytical framework for their study. In particular, the following characterization theorem discovered and proved by Shary is of central importance. Its statement differs from the one proposed in [28] only by the new convention chosen for the relation between the quantifiers and the proper/improper qualities of generalized intervals. Also, the proof proposed here is reproduced from the one given in [28] using the new convention. This proof provides a interesting insight of the relationship between the group structure of the generalized interval addition and its interpretation in the context of AE-solution sets.

Theorem 1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$. *Then*

$$x \in \Sigma(\mathbf{A},\mathbf{b}) \iff (\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}. \tag{16}$$

Proof. By definition of an AE-solution set, $x \in \Sigma(\mathbf{A},\mathbf{b})$ is equivalent to

$$\begin{aligned}\forall A^\forall \in \mathbf{A}^\forall, \forall b^\forall \in \mathbf{b}^\forall, \exists A^\exists \in \mathbf{A}^\exists, \exists b^\exists \in \mathbf{b}^\exists,\\ (A^\forall + A^\exists)x = b^\forall + b^\exists.\end{aligned} \tag{17}$$

where $\mathbf{A}^\forall$, $\mathbf{A}^\forall$, $\mathbf{b}^\forall$ and $\mathbf{b}^\forall$ are defined as in (9) and (10), so $\mathbf{A} = \mathbf{A}^\exists + \mathrm{dual}\,\mathbf{A}^\forall$ and $\mathbf{b} = \mathbf{b}^\exists + \mathrm{dual}\,\mathbf{b}^\forall$. Now, $(A^\forall + A^\exists)x = (b^\forall + b^\exists)$ is equivalent to $A^\forall x - b^\forall = -A^\exists x + b^\exists$. Furthermore, obviously both $\mathbf{A}^\forall x - \mathbf{b}^\forall = \{A^\forall x - b^\forall | A^\forall \in \mathbf{A}^\forall, b^\forall \in \mathbf{b}^\forall\}$ and $-\mathbf{A}^\exists x + \mathbf{b}^\exists = \{-A^\exists x + b^\exists | A^\exists \in \mathbf{A}^\exists, b^\exists \in \mathbf{b}^\exists\}$ hold because each parameter has only one occurrence in the whole system. As a consequence, (17) is equivalent to

$$\mathbf{A}^\forall x - \mathbf{b}^\forall \subseteq -\mathbf{A}^\exists x + \mathbf{b}^\exists. \tag{18}$$

Now, adding $\mathrm{dual}\,(\mathbf{A}^\exists x) + \mathrm{dual}\,\mathbf{b}^\forall$ and using the group property of the generalized interval addition, one proves that (18) is equivalent to

$$\mathbf{A}^\forall x + \mathrm{dual}\,(\mathbf{A}^\exists x) \subseteq \mathbf{b}^\exists + \mathrm{dual}\,\mathbf{b}^\forall. \tag{19}$$

Finally, noticing that $\mathbf{A}^\forall x + \mathrm{dual}\,(\mathbf{A}^\exists x) = \mathbf{A}^\forall x + (\mathrm{dual}\,\mathbf{A}^\exists)x = (\mathbf{A}^\forall + \mathrm{dual}\,\mathbf{A}^\exists)x$, the last equality being a consequence of the distributivity of the addition and multiplication in $\mathbb{KR}$, one obtain the equivalent inclusion $(\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}$. □

The general characterization (16) is an elegant generalization of the previously known characterizations in the special cases of united, tolerable and controllable solution sets:

- If $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and $\mathbf{b} \in \mathbb{IR}^n$ then $(\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}$ is equivalent to $\mathbf{A}x \cap \mathbf{b} \neq \emptyset$ which is the well-known characterization of united solution sets.
- If $\mathbf{A} \in \overline{\mathbb{IR}}^{n\times n}$ and $\mathbf{b} \in \mathbb{IR}^n$ then $(\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}$ is equivalent to $(\mathrm{pro}\,\mathbf{A})x \subseteq \mathbf{b}$ which is the well-known characterization of tolerable solution sets.
- If $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and $\mathbf{b} \in \overline{\mathbb{IR}}^n$ then $(\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}$ is equivalent to $(\mathrm{pro}\,\mathbf{b}) \subseteq \mathbf{A}x$ which is the well-known characterization of controllable solution sets.

As a direct consequence of Theorem 1, we have $\mathbf{A} \subseteq \mathbf{A}'$ and $\mathbf{b} \subseteq \mathbf{b}'$ imply $\Sigma(\mathbf{A},\mathbf{b}) \subseteq \Sigma(\mathbf{A}',\mathbf{b}')$. In particular $\Sigma(\mathbf{A},\mathbf{b}) \subseteq \Sigma(\mathrm{pro}\,\mathbf{A}, \mathrm{pro}\,\mathbf{b})$ which states that universal quantifiers are more restrictive than existential ones.

4 Approximation of linear AE-solution sets

As in the classical interval theory for the approximation of linear united solution sets, the approximation of linear AE-solution sets can be done both through the characterization of the solutions of some auxiliary interval equation or using contracting interval operators. The following subsections give a survey of these methods. The use of the new convention relating the quantifiers and the proper/improper quality of generalized intervals allows us to give an homogenized presentation with the classical interval framework.

Throughout this section, $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and $\mathbf{b} \in \mathbb{KR}^n$ are considered. They correspond to a linear AE-solution set $\Sigma(\mathbf{A},\mathbf{b})$.

4.1 Auxiliary interval equations

This section presents three generalized interval equations whose solutions can be interpreted as some approximation of linear AE-solution sets. The technique which consists in solving some auxiliary interval equation to build approximation of AE-solution set is called the *formal algebraic approach* to AE-solution set approximation (see [24, 13, 23, 14, 26, 21, 22, 3]).

4.1.1 Auxiliary interval equation dedicated to inner approximation

While inner approximation is not studied in the classical interval analysis of linear united solution sets, it arises naturally in the generalized interval analysis of linear AE-solution sets. The following theorem provides a way to build an inner approximation of some linear AE-solution set. The proof is reproduced from the one given by Shary in [28] but using the new convention relating the quantifiers and the proper/improper quality of generalized intervals.

Theorem 2. *Every proper solution* $\mathbf{x} \in \mathbb{IR}^n$ *of the interval equation*

$$(\mathrm{dual}\,\mathbf{A})\mathbf{x} = \mathbf{b} \tag{20}$$

is an inner approximation of $\Sigma(\mathbf{A},\mathbf{b})$, *i.e.* $\mathbf{x} \subseteq \Sigma(\mathbf{A},\mathbf{b})$.

Proof. Let a proper interval vector $\mathbf{x}$ be a formal solution of the equation (20) and $\tilde{x} \in \mathbf{x}$. Then, in view of the monotonicity of the Kaucher arithmetic, we have $(\mathrm{dual}\,\mathbf{A})\tilde{x} \subseteq (\mathrm{dual}\,\mathbf{A})\mathbf{x} = \mathbf{b}$, that is $x \in \Sigma(\mathbf{A},\mathbf{b})$ by Theorem 1. □

Remark. Using the convention relating the quantifiers and the proper/improper quality of generalized intervals previously used in [28, 21, 3], the equation (20) is written $\mathbf{A}\mathbf{x} = \mathbf{b}$ in the latter papers. However, Kupriyanova gave in [13] the same expression and the same interpretation for the equation (20) but in the restricted case of united solution sets.

4.1.2 Auxiliary interval equations dedicated to outer approximation

Two generalized interval equations have been proposed for the outer approximation of linear AE-solution sets. The following theorem is proposed by Shary in [26] (see also [28]).

Theorem 3. *Suppose that* $\rho(|I - \mathrm{pro}\,\mathbf{A}|) < 1$. *Then, the following interval equation has an unique solution* $\mathbf{x} \in \mathbb{KR}^n$*:*

$$\mathbf{x} = (I - \mathbf{A})\mathbf{x} + \mathbf{b}. \tag{21}$$

Furthermore, if the solution $\mathbf{x}$ *is proper then it contains* $\Sigma(\mathbf{A}, \mathbf{b})$*; if the solution* $\mathbf{x}$ *is not proper then* $\Sigma(\mathbf{A}, \mathbf{b})$ *is empty.*

Equation (21) is written $\mathbf{x} = (I - \mathrm{dual}\,\mathbf{A})\mathbf{x} + \mathbf{b}$ in [28, 21, 3]. Thanks for the newly introduced convention relating the quantifiers and the proper/improper quality of generalized intervals, one can now see the similitude of this equation with the Krawczyk operator dedicated to linear united solution sets: in the case where $\mathbf{A}$ and $\mathbf{b}$ are proper, iterating the fixed point form of Equation (21) corresponds to the Krawczyk operator where the intersection with the last approximation would not be performed.

4.1.3 Solving generalized interval equations

Several ways can be used to solve the auxiliary interval equations presented in the previous section. The most simple, though very efficient, is to write them in a fixed point form and to iterate this fixed point form. If this iteration converges then its limit is a solution of the fixed point form, and hence of the original equation. A widely used fixed point form of (20) is

$$\bigwedge_{i\in\{1,\dots,n\}} \mathbf{x}_i = \frac{1}{\mathrm{dual}\,\mathbf{A}_{ii}}\Big(\mathbf{b}_i - \sum_{\substack{j\in[1..n]\\ j\neq i}} \mathrm{dual}\,(\mathbf{A}_{ij}\mathbf{x}_j)\Big). \tag{22}$$

and is defined provided that $0 \notin \mathrm{pro}\,\mathbf{A}_{ii}$ for all $i \in \{1,\dots,n\}$. It is proved in [3] that this fixed point iteration converges to the unique solution of (20) if $\mathrm{pro}\,\mathbf{A}$ is an H-matrix (see also [14]). Equation (21) is already in fixed point form. The fixed point iteration is proved to converge to the unique solution of (21) provided that $\rho(|I - \mathrm{pro}\,\mathbf{A}|) < 1$ (see [26, 28]).

Other resolution processes have been proposed: Shary has proposed in [23, 25] to immerse the interval equation to be solved in $\mathbb{R}^{2n}$ where a system of $2n$ real equations is obtained making explicit the expression of the generalized interval arithmetic. The resulting real system is not differentiable and a dedicated solving process have been proposed by Shary. Also, Sainz et al. have proposed in [21] to change the auxiliary interval equation to an optimization problem which can be handled using existing optimization softwares. Finally, the group structures of the generalized interval arithmetic allow in some situation to solve equations and systems of equations. This has been studied e.g. in [19].

4.2 Generalized interval operators

Only one generalized interval operator is available for the outer approximation of linear AE-solution sets: Shary has proposed a generalized interval Gauss-Seidel operator in [27] (see also [28]). It has the same interpretation as its classical interval counterpart, but extended to linear AE-solution sets. Using the newly introduced convention relating the quantifiers and the proper/improper quality of generalized intervals, Shary's generalized interval Gauss-Seidel operator now has the same expression as its classical counterpart.

Theorem 4. *Provided that* $0 \notin \mathrm{pro}\,\mathbf{A}_{ii}$ *for all* $i \in \{1,\dots,n\}$*, the generalized interval Gauss-Seidel operator* $\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})$ *is defined by*

$$\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})_i := \frac{1}{\mathbf{A}_{ii}}\Big(\mathbf{b}_i - \sum_{j<i} \mathbf{A}_{ij}\mathbf{y}_j - \sum_{j>i} \mathbf{A}_{ij}\mathbf{x}_j\Big), \tag{23}$$

If $\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})$ *is proper then* $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x} \subseteq \Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})$. *Otherwise* $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x} = \emptyset$.

Remark. Using the other convention relating the quantifiers and the proper/improper quality of generalized intervals, the generalized interval Gauss-Seidel operator is written $(1/\mathrm{dual}\,\mathbf{A}_{ii})\big(\mathbf{b}_i - \sum_{j<i}(\mathrm{dual}\,\mathbf{A}_{ij})\mathbf{y}_j - \sum_{j>i}(\mathrm{dual}\,\mathbf{A}_{ij})\mathbf{x}_j\big)$ in [28].

Theorem 4 is the typical situation where the newly introduced convention show its usefulness: the classical and generalized interval Gauss-Seidel operators now share both their expression and their interpretation. Note however that the classical interval Gauss-Seidel operator can be applied in situations where $0 \in \mathbf{A}_{ii}$ thanks to the use of an extended division while the generalized interval Gauss-Seidel operator proposed by Shary needs $0 \notin \mathbf{A}_{ii}$ for all $i \in \{1,\dots,n\}$. This gap will be filled in Section 6.

4.3 Preconditioning

Preconditioning consists in studying an auxiliary solution set where the approximation methods both keep their interpretation and have a better behavior. Two kinds of preconditioning have been proposed for linear AE-solution sets. First, Shary has proved that

$$\Sigma(\mathbf{A}, \mathbf{b}) \subseteq \Sigma(C\mathbf{A}, C\mathbf{b}), \tag{24}$$

where $C \in \mathbb{R}^{n\times n}$ is regular, hence generalizing the preconditioning process used in the context of united solution sets.

Indeed by (24), an outer approximation of $\Sigma(C\mathbf{A}, C\mathbf{b})$ is also an outer approximation of $\Sigma(\mathbf{A}, \mathbf{b})$. While such a preconditioning process improves the stability of the Gauss-Seidel operator, it is even more important when using Theorem 3 where the condition $\rho(|I - \mathrm{pro}\,\mathbf{A}|) < 1$ is actually very restrictive. Using the midpoint inverse preconditioning, i.e. $C = (\mathrm{mid}\,\mathbf{A})^{-1}$, Theorem 3 can be applied with any interval matrix that satisfies $\mathrm{pro}\,\mathbf{A}$ is strongly regular (see [28] for more details).

Another preconditioning process dedicated to inner approximation has been proposed by the first author in [3]. There, the following inclusion is proved[3]:

$$\{\tilde{x}+Cu | u \in \Sigma(\mathrm{imp}\,\mathbf{A}C, b-\mathrm{dual}\,\mathbf{A}\tilde{x})\} \subseteq \Sigma(\mathbf{A}, \mathbf{b}), \quad (25)$$

where $C \in \mathbb{R}^{n\times n}$ is regular. As a consequence, if $\mathbf{u}$ is an inner approximation of the auxiliary AE-solution set $\Sigma(\mathrm{imp}\,\mathbf{A}C, b - \mathrm{dual}\,\mathbf{A}\tilde{x})$ then the following inclusion holds:

$$\{\tilde{x} + Cu | u \in \mathbf{u}\} \subseteq \Sigma(\mathbf{A}, \mathbf{b}). \quad (26)$$

Therefore, computing an inner approximation $\mathbf{u}$ of the auxiliary AE-solution set $\Sigma(\mathrm{imp}\,\mathbf{A}C, b - \mathrm{dual}\,\mathbf{A}\tilde{x})$ gives rise to an inner approximation of the original AE-solution set $\Sigma(\mathbf{A}, \mathbf{b})$ under the form of a parallelepiped $\{\tilde{x} + Cu | u \in \mathbf{u}\}$. Inner approximation methods (in particular the fixed point iteration) often have a better behavior when applied to $\Sigma(\mathrm{imp}\,\mathbf{A}C, b - \mathrm{dual}\,\mathbf{A}\tilde{x})$ using e.g. $C = (\mathrm{mid}\,\mathbf{A})^{-1}$.

5 Generalized interval Krawczyk operator

Thank to the new conventions used in the present paper, it is now clear that the auxiliary interval equation (21) is similar to the Krawczyk operator for linear united solution sets. It is therefore natural to introduce a generalized interval Krawczyk operator.

Theorem 5. *The generalized interval Krawczyk operator is defined by*

$$\mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x}) := \Big((I - \mathbf{A})\mathbf{x} + \mathbf{b}\Big) \wedge \mathbf{x}, \quad (27)$$

where an non-proper interval vector is interpreted as the emptyset. If $\mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$ is proper then $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x} \subseteq \mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$. Otherwise $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x} = \emptyset$.

Proof. Consider any $x \in \mathbf{x} \cap \Sigma(\mathbf{A}, \mathbf{b})$. Then by Theorem 1 the inclusion $(\mathrm{dual}\,\mathbf{A})x \subseteq \mathbf{b}$ holds. The generalized interval distributivity gives rise to $(\mathrm{dual}\,\mathbf{A})x = x + (\mathrm{dual}\,\mathbf{A} - I)x$. Therefore $x + (\mathrm{dual}\,\mathbf{A} - I)x \subseteq \mathbf{b}$ holds. Now, adding $-\mathrm{dual}\,\big((\mathrm{dual}\,\mathbf{A} - I)x\big) = (I - \mathbf{A})x$ to each side of the inclusion gives rise to $x \subseteq (I - \mathbf{A})x + \mathbf{b}$. Finally the interval inclusion isotonicity proves $x \subseteq (I - \mathbf{A})\mathbf{x} + \mathbf{b}$ and hence $x \subseteq \mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$ because $x \subseteq \mathbf{x}$. As a consequence, if $\mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$ is proper then $x \in \mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$. On the other hand if $\mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$ is not proper then no x can satisfy $x \subseteq \mathbf{K}(\mathbf{A}, \mathbf{b}, \mathbf{x})$ and therefore $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x}$ is empty. □

[3] The inclusion (25) is actually an obvious consequence of Proposition 4.1 and Proposition 4.2 of [3]

The generalized interval Krawczyk operator present several advantages on its generalized interval equation counterpart (21): first it can be applied with no restriction on $\mathbf{A}$. Second, an initial enclosure can be provided in order to enclose $\Sigma(\mathbf{A}, \mathbf{b}) \cap \mathbf{x}$. If no initial enclosure is available and if $\mathbf{A}$ is strongly regular then one can use $[-1, 1]\,\langle \mathrm{pro}\,\mathbf{A}\rangle^{-1}\,|\mathrm{pro}\,\mathbf{b}|$ which is a enclosure of $\Sigma(\mathrm{pro}\,\mathbf{A}, \mathrm{pro}\,\mathbf{b})$ (cf. [15]).

6 Improved generalized interval Gauss-Seidel operator

The generalized interval Gauss-Seidel (IGS in this section) operator presented in Section 4.2 cannot be applied if $0 \in \mathrm{pro}\,\mathbf{A}_{ii}$ for some i. The situation was similar with the first version of classical IGS operator proposed in [20]. Latter, an improved classical IGS have been proposed in [7] which allows dealing with zero-containing intervals. This section presents an improved version of the generalized interval Gauss-Seidel operator that can deal with any generalized interval matrix, hence extending the ideas of [7] to linear AE-solution sets.

6.1 One dimensional case

Following the presentation given in [15] in the context of united solution sets, the improved generalized IGS operator is first defined in the case $n = 1$. In this situation, the AE-solution set $\Sigma(\mathbf{a}, \mathbf{b})$ is defined by two generalized intervals $\mathbf{a}, \mathbf{b} \in \mathbb{KR}$. Given furthermore a proper interval $\mathbf{x} \in \mathbb{IR}$, the improved generalized IGS operator is defined as

$$\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) := \square\big(\Sigma(\mathbf{a}, \mathbf{b}) \bigcap \mathbf{x}\big). \quad (28)$$

The next proposition provides a computable expression of $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$.

Proposition 1. *In the following expressions, an improper interval is interpreted as an emptyset.*

1. *If* $0 \notin \mathrm{pro}\,\mathbf{a}$ *then* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) = (\mathbf{b}/\mathbf{a}) \bigwedge \mathbf{x}$.
2. *If* $0 \subseteq \mathbf{a}$ *then define* E *as the open interval* $\big]\min\{0, \underline{\mathbf{b}}/\underline{\mathbf{a}}, \overline{\mathbf{b}}/\overline{\mathbf{a}}\}\ ,\ \max\{0, \underline{\mathbf{b}}/\overline{\mathbf{a}}, \overline{\mathbf{b}}/\underline{\mathbf{a}}\}\big[$[4]. *Then,* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) = \square(\mathbf{x}\backslash E)$.

[4] This open interval is well defined because the first bound is always lower or equal than the second. Note furthermore that $]0, 0[= \emptyset$ while $\square\emptyset = \emptyset$.

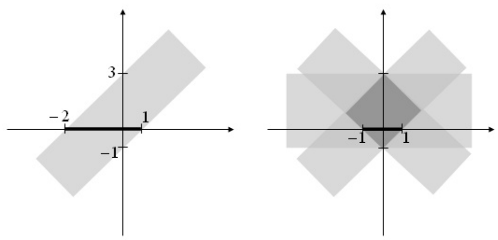

Figure 1.

3. *If* $\mathbf{a} \subseteq 0$ *then* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) =$
$\big[\max\{\underline{\mathbf{b}}/\underline{\mathbf{a}}, \overline{\mathbf{b}}/\overline{\mathbf{a}}\}\ ,\ \min\{\underline{\mathbf{b}}/\overline{\mathbf{a}}, \overline{\mathbf{b}}/\underline{\mathbf{a}}\}\big] \bigwedge \mathbf{x}$.

Proof. Cf. [4] pages 105-107. □

Remark. As in [15], the expression of $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is not detailed in the cases where $\underline{\mathbf{a}} = 0$ or $\overline{\mathbf{a}} = 0$. Also, one can check that when $0 \in \mathbf{a}$ and $\mathbf{b}$ is proper, the expression proposed here coincides with the one proposed in [15] in the context of one dimensional linear united solution sets.

The next example illustrates Proposition 1.

Example 1. Consider $\mathbf{a} = [1, -1]$ and $\mathbf{b} = [-1, 3]$ and the one dimensional AE-solution set $\Sigma(\mathbf{a}, \mathbf{b}) =$

$$\big\{x \in \mathbb{R} \mid \big(\forall a \in [-1, 1]\big)\big(\exists b \in [-1, 3]\big)\big(ax = b\big)\big\}\,. \quad (29)$$

Determining this simple AE-solution set can be done graphically. Fix a value of a in $[-1, 1]$. Then the set $\big\{x \in \mathbb{R} \mid \big(\exists b \in [-1, 3]\big)\big(ax = b\big)\big\}$ can be easily determined: it is the set of solutions of the equation $ax = b$ for some $b \in [1, 3]$ (it is plotted on the left hand side graphic of Figure 1 for $a = 1$). Therefore, in order to obtain the set of $x \in \mathbb{R}$ that satisfies $\big(\exists b \in [-1, 3]\big)\big(ax = b\big)$ for any $a \in [-1, 1]$, it remains to intersect all the strips obtained for $a \in [-1, 1]$. Three of these strips and the resulting intersection are plotted on the right hand side graph where we can see that $\Sigma(\mathbf{a}, \mathbf{b}) = [-1, 1]$. It can be noted that the AE-solution set is bounded while $0 \in \mathrm{pro}\ \mathbf{a}$. Now, given an initial $\mathbf{x} = [-\infty, \infty]$, the generalized IGS $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$ leads to $\big[\max\{\underline{\mathbf{b}}/\underline{\mathbf{a}}, \overline{\mathbf{b}}/\overline{\mathbf{a}}\}\ ,\ \min\{\underline{\mathbf{b}}/\overline{\mathbf{a}}, \overline{\mathbf{b}}/\underline{\mathbf{a}}\}\big]$, which is equal to $[-1, 1]$. This is indeed equal to $\Sigma(\mathbf{a}, \mathbf{b})$.

Link with a generalized interval division

Proposition 1 can be used to extend the generalized interval division to cases where the numerator's proper projection contains zero: in these cases $\mathbf{b}/\mathbf{a}$ is defined by

$$\begin{matrix}]-\infty, \min\{0, \underline{\mathbf{b}}/\underline{\mathbf{a}}, \overline{\mathbf{b}}/\overline{\mathbf{a}}\}] \\ \bigcup\ [\max\{0, \underline{\mathbf{b}}/\overline{\mathbf{a}}, \overline{\mathbf{b}}/\underline{\mathbf{a}}\}, +\infty[\end{matrix} \quad \text{if } 0 \subseteq \mathbf{a} \quad (30)$$

$$\big[\max\{\underline{\mathbf{b}}/\underline{\mathbf{a}}, \overline{\mathbf{b}}/\overline{\mathbf{a}}\}\ ,\ \min\{\underline{\mathbf{b}}/\overline{\mathbf{a}}, \overline{\mathbf{b}}/\underline{\mathbf{a}}\}\big] \quad \text{if } \mathbf{a} \subseteq 0. \quad (31)$$

Then, the generalized IGS expression can be written $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) = \square(1/\mathbf{a} \cap \mathbf{x})$, where an improper $1/\mathbf{a}$ is interpreted as an empty set for the intersection with $\mathbf{x}$. An other extension of the generalized interval division has been proposed in [10] and further studied in [18]. The study of the relationship between these two extensions remains to be conducted.

6.2 The general case

A n dimensional generalized IGS can now be constructed: consider $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{b} \in \mathbb{KR}^n$ and $\mathbf{x} \in \mathbb{IR}^n$. Then the generalized IGS $\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})$ is defined in the following way: for $i \in \{1, \ldots, n\}$

$$\Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x})_i := \Gamma\Big(\mathbf{A}_{ii}\ ,\ \mathbf{b}_i - \sum_{j<i} \mathbf{A}_{ij}\mathbf{y}_j - \sum_{j>i} \mathbf{A}_{ij}\mathbf{x}_j\ ,\ \mathbf{x}_i\Big), \quad (32)$$

where $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is defined in Subsection 6.1. The next theorem provides the interpretation of this improved generalized IGS.

Theorem 6. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{b} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{x} \in \mathbb{IR}^{n\times n}$. *Then*

$$\Sigma(\mathbf{A}, \mathbf{b}) \bigcap \mathbf{x} \subseteq \Gamma(\mathbf{A}, \mathbf{b}, \mathbf{x}) \subseteq \mathbf{x} \quad (33)$$

Proof. Cf. [4] pages 107-108. □

The usefulness of the improved generalized IGS operator is illustrated by the following example. As in [5], both interval Gauss-Seidel operators are applied not only on the diagonal entries of the matrix but on all entries, leading to $n \times n$ applications of the one dimensional operator instead of only n applications.

Example 2. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & [-1, 1] \\ 2 & [1, -1] \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} [1, -1] \\ [-1, 1] \end{pmatrix}. \quad (34)$$

The AE-solution set $\Sigma(\mathbf{A}, \mathbf{b})$ is displayed on Figure 2. Note that this AE-solution set is bounded while $\mathrm{pro}\ \mathbf{A}$ is not regular which is not possible in the context of united solution sets. The initial box is set to $([-10, 10], [0, 20])$. The interval gauss Seidel-Seidel without the improvement proposed in this paper leads to the contracted box $([-1, 1], [0, 20])$. The improved Gauss-Seidel leads to $([-1, 1], [0, 4])$ and hence provide an additional contraction.

7 Conclusion

AE-solution sets generalize the united-solution sets allowing different quantifications of parameters, while universal quantifiers precede existential ones. They allow to

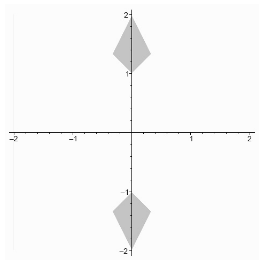

Figure 2.

model useful situations. It has become clear that generalized intervals are the right framework for the study of AE-solution sets. Thanks to the proposition of a new convention for the association of a quantifier to a generalized interval, the studies of united solution sets and AE-solution sets have been homogenized. This has allowed to propose a generalized interval Krawczyk operator and an extension of Shary's generalized interval Gauss-Seidel to interval matrices with zero-containing intervals on the diagonal.

The interval Gauss elimination and Hansen-Bliek algorithm have not been considered in this paper. They have have been investigated by the authors in [6] and [1] respectively. However, their generalization to AE-solution sets is not as direct as the techniques presented in the present paper, and still some work remains to be conducted in this direction.

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