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A Right-Preconditioning Process for the Formal-Algebraic Approach to Inner and Outer Estimation of AE-Solution Sets

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Abstract. A right-preconditioning process for linear interval systems has been presented by Neumaier in 1987. It allows the construction of an outer estimate of the united solution set of a square linear interval system in the form of a parallelepiped. The denomination "right-preconditioning" is used to describe the preconditioning processes which involve the matrix product $\mathbf{A}C$ in contrast to the (usual) left-preconditioning processes which involve the matrix product $C\mathbf{A}$, where $\mathbf{A}$ and C are respectively the interval matrix of the studied linear interval system and the preconditioning matrix.

The present paper presents a new right-preconditioning process similar to the one presented by Neumaier in 1987 but in the more general context of the inner and outer estimations of linear AE-solution sets. Following the spirit of the formal-algebraic approach to AE-solution sets estimation, summarized by Shary in 2002, the new right-preconditioning process is presented in the form of two new auxiliary interval equations. Then, the resolution of these auxiliary interval equations leads to inner and outer estimates of AE-solution sets in the form of parallelepipeds. This right-preconditioning process has two advantages: on one hand, the parallelepipeds estimates are often more precise than the interval vectors estimates computed by Shary. On the other hand, in many situations, it simplifies the formal algebraic approach to inner estimation of AE-solution sets. Therefore, some AE-solution sets which were almost impossible to inner estimate with interval vectors, become simple to inner estimate using parallelepipeds. These benefits are supported by theoretical results and by some experimentations on academic examples of linear interval systems.

1. Introduction

A well studied problem in interval analysis is the computation of a superset of the following united solution set:

$$\Sigma(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \mathbf{A},\ \exists b \in \mathbf{b},\ Ax = b\},$$

where $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is an interval matrix and $\mathbf{b} \in \mathbb{IR}^n$ an interval vector. Many methods are available which build such a superset in the form of an interval vector—Krawczyk, Gauss-Seidel, interval Gauss elimination, etc. (see [10]). Preconditioning a united solution set consists in considering an auxiliary united solution set on

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which the method used to build an outer estimate has a better behavior. The mostly used preconditioning is the left-preconditioning* which consists in studying the following auxiliary united solution set:

$$\Sigma(\mathbf{A}', \mathbf{b}'),$$

where $\mathbf{A}' = C\mathbf{A}$ and $\mathbf{b}' = C\mathbf{b}$ with $C \in \mathbb{R}^{n\times n}$ a nonsingular real matrix. This left-preconditioned system is adequate for outer estimation because of the following inclusion:

$$\Sigma(\mathbf{A}, \mathbf{b}) \subseteq \Sigma(\mathbf{A}', \mathbf{b}').$$

Therefore an interval vector $\mathbf{x}$ which contains the left-preconditioned united solution set also contains the original one:

$$\Sigma(\mathbf{A}', \mathbf{b}') \subseteq \mathbf{x} \implies \Sigma(\mathbf{A}, \mathbf{b}) \subseteq \mathbf{x}.$$

A wildly used preconditioning matrix is the midpoint inverse of $\mathbf{A}$, i.e. $C = (\operatorname{mid} \mathbf{A})^{-1}$. In the cases where the interval coefficients of $\mathbf{A}$ have small enough radius, the left preconditioned interval matrix $\mathbf{A}' = (\operatorname{mid} \mathbf{A})^{-1}\mathbf{A}$ is close to the identity matrix, and the methods for constructing interval vector outer estimates have a good behavior on the left-preconditioned system—see [10] for a detailed presentation of left-preconditioned of united solution sets.

Right-preconditioning is an alternative to left-preconditioning which was proposed in [12] in the context of outer estimation of united solution sets. It consists in considering an auxiliary united solution set where the product $\mathbf{A}C$, instead of $C\mathbf{A}$, is involved. It is proved in [12] that considering a right-preconditioned united solution set gives rise to a superset in the form of a parallelepiped, also called a *skew box*. It is also proved in [12] that estimation with parallelepipeds can be much more accurate than with interval vectors.

1.1. AE-Solution Sets

The united solution set of an interval system has been generalized to AE-solution sets in [16], allowing the distinction between universally quantified parameters and existentially quantified parameters. Given an interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and an interval vector $\mathbf{b} \in \mathbb{IR}^n$, an AE-solution set is denoted by $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$, where α and β stand for a description of the quantification associated to the parameters a_{ij} and b_k. The following specific cases of AE-solution sets are usually met:

- united solution set: $\Xi_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \mathbf{A},\ \exists b \in \mathbf{b},\ Ax = b\}$;
- tolerable solution set: $\Xi_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid \forall A \in \mathbf{A},\ \exists b \in \mathbf{b},\ Ax = b\}$;
- controllable solution set: $\Xi_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid \forall b \in \mathbf{b},\ \exists A \in \mathbf{A},\ Ax = b\}$.

* The “left-preconditioning” is usually simply called “preconditioning.” However, this precision will allow to distinguish the left-preconditioning and the right-preconditioning.

The general definition of AE-solution sets, which allows arbitrary quantification of the parameters, is given in Section 2. If the inner estimation of united solution sets is usually not considered in the classical interval theory, both inner and outer estimations of AE-solution sets are considered in a unified framework.

The estimation of AE-solution sets shall benefit of a preconditioning process. Both interval parameters and quantifications can be changed to obtain a preconditioned AE-solution set: preconditioning the AE-solution set $\Xi_{\alpha,\beta}(\mathbf{A}, \mathbf{b})$ can lead to the auxiliary AE-solution set $\Xi_{\alpha'\beta'}(\mathbf{A}', \mathbf{b}')$. It was noticed in [16] that the "naively left-preconditioned" AE-solution set defined by

$$\alpha' = \alpha, \qquad \beta' = \beta, \qquad \mathbf{A}' = C\mathbf{A}, \qquad \mathbf{b}' = C\mathbf{b}$$

is not compatible with the outer estimation of AE-solution sets because in general the inclusion $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}) \subseteq \Xi_{\alpha\beta}(C\mathbf{A}, C\mathbf{b})$ does not hold. An original left-preconditioning process that is compatible with outer estimation of AE-solution sets was proposed in [19]. Thanks to this preconditioning, the scope of outer-estimation of AE-solution sets has been enlarged to any AE-solution set involving strongly regular interval matrices, which are likely to be met within applications.

Up to now, no preconditioning process was available for inner estimation. Inner estimation indeed needs a specific preconditioning process because the preconditioned AE-solution set has to be included inside the original one, i.e.

$$\Xi_{\alpha'\beta'}(\mathbf{A}', \mathbf{b}') \subseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}),$$

where $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ and $\Xi_{\alpha'\beta'}(\mathbf{A}', \mathbf{b}')$ are respectively the original and the preconditioned AE-solution sets. As a consequence,

$$\mathbf{x} \subseteq \Xi_{\alpha'\beta'}(\mathbf{A}', \mathbf{b}') \implies \mathbf{x} \subseteq \Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b}).$$

That is, an inner estimate of the preconditioned AE-solution will also be an inner estimate of the original AE-solution set. The argumentation presented in [16] also proves that the "naively left-preconditioned" AE-solution set is not compatible with inner estimation, even if it was not emphasized.

1.2. Contribution

The main contribution of the paper is a right-preconditioning process dedicated to inner and outer estimation of AE-solution sets. Like in the context of united solution sets, where the right-preconditioning process proposed in [12] leaded to supersets in the form of skew boxes, the proposed right-conditioning process allows the construction of inner and outer estimates in the form of skew boxes. This is illustrated by Figure 1 where are represented an AE-solution set $\Xi(\mathbf{A}, \mathbf{b})$ together with its inner and outer skew box estimates $\mathbf{s}$ and $\mathbf{s}'$.

The advantages of this right-preconditioned process are investigated theoretically and through its application to academic examples of linear interval systems.

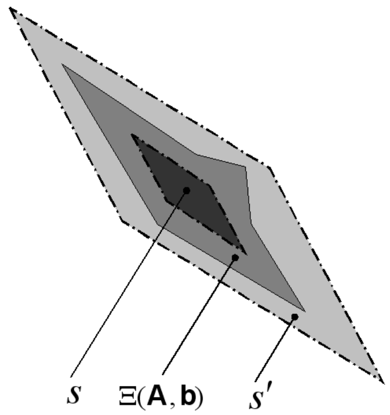

Figure 1. An AE-solution set with its inner and outer skew box estimates.

1.3. OUTLINE OF THE PAPER

Generalized intervals and their use in the context of the estimation of AE-solution sets are presented in Section 2. In particular, the formal-algebraic approach is presented in that section. An improved sufficient condition for the uniqueness of the solution of the interval equation $\mathbf{Ax} = \mathbf{b}$ is presented in Section 3. The right-preconditioning process for inner and outer estimation of AE-solution sets is presented in Section 4. Some general statements about the quality of this right-preconditioning process are presented in Section 5. Finally, some experimentations conducted on academic examples of linear interval systems are presented in Section 6.

2. The Formal-Algebraic Approach to AE-Solution Sets Estimation

The formal-algebraic approach to the estimation of AE-solution sets was proposed in the 90's—see papers by Shary [16]–[18], Markov [8], Sainz [14], [15] and extensive references. This approach consists in providing both auxiliary interval equations in the space of generalized intervals and the way their solutions are linked to the studied AE-solution set. Then, some dedicated algorithms are used so as to compute the solution of these auxiliary interval equations and hence, estimate the studied AE-solution set.

2.1. GENERALIZED INTERVALS

Generalized intervals are intervals whose bounds are not constrained to be ordered, for example $[-1, 1]$ and $[1, -1]$ are generalized intervals. They have been introduced in [5], [6] so as to improve both the algebraic structure and the order structure of the classical intervals. Generalized intervals are written as closed intervals, even if no open generalized intervals are considered. The set of generalized intervals is denoted by $\mathbb{KR}$ and is decomposed into three subsets:

- The set of proper intervals whose bounds are ordered increasingly. These proper intervals are identified with classical intervals. The set of proper intervals is denoted by the same symbol as the one used for classical intervals, i.e. $\mathbb{IR} = \{[a, b] \mid a \leq b\}$.
- The set of improper intervals whose bounds are ordered decreasingly. It is denoted by $\overline{\mathbb{IR}} = \{[a, b] \mid a \geq b\}$.
- The set of degenerated intervals $[a, a]$, where $a \in \mathbb{R}$, which are both proper and improper. A degenerated interval will also be denoted by the corresponding real a.

Therefore, from a set of reals $\{x \in \mathbb{R} \mid a \leq x \leq b\}$, one can build the two generalized intervals $[a, b]$ and $[b, a]$. It will be useful to change one to the other keeping the underlying set of reals unchanged using the following operations:

- the dual operation: $\operatorname{dual}[a, b] = [b, a]$;
- the proper projection: $\operatorname{pro}[a, b] = [\min\{a, b\}, \max\{a, b\}]$;
- the improper projection: $\operatorname{imp}[a, b] = [\max\{a, b\}, \min\{a, b\}]$.

The operations mid, rad, and $|\,.\,|$ are defined as in the case of classical intervals:

- $\operatorname{mid}[a, b] = \frac{a+b}{2}$;
- $\operatorname{rad}[a, b] = \frac{b-a}{2}$;
- $|[a, b]| = \max\{|a|, |b|\}$.

It can be noticed that proper intervals have positive radius whereas improper intervals have negative radius—degenerated intervals have a null radius. The generalized intervals are partially ordered by an inclusion which prolongates the inclusion of classical intervals. Given two generalized intervals $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $\mathbf{y} = [\underline{\mathbf{y}}, \overline{\mathbf{y}}]$, the inclusion is defined by

$$\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \leq \underline{\mathbf{x}} \wedge \overline{\mathbf{x}} \leq \overline{\mathbf{y}}.$$

This inclusion is related to the dual operation in the following way:

$$\mathbf{x} \subseteq \mathbf{y} \iff (\operatorname{dual}\mathbf{x}) \supseteq (\operatorname{dual}\mathbf{y}).$$

The so-called Kaucher arithmetic extends the classical interval arithmetic. Its definition can be found in [5], [6], and [16]. When it is not misunderstanding, the Kaucher multiplication will be denoted by $\mathbf{x}\,\mathbf{y}$ instead of $\mathbf{x} \times \mathbf{y}$. The Kaucher arithmetic has better algebraic properties than the classical interval arithmetic: The Kaucher addition is a group. The opposite of an interval $\mathbf{x}$ is $-\operatorname{dual}\mathbf{x}$, i.e.

$$\mathbf{x} + (-\operatorname{dual}\mathbf{x}) = \mathbf{x} - \operatorname{dual}\mathbf{x} = [0, 0].$$

The Kaucher multiplication restricted to generalized intervals whose proper projection does not contains 0 is also a group. The inverse of such a generalized interval $\mathbf{x}$ is $1 / (\text{dual}\,\mathbf{x})$, i.e.

$$\mathbf{x} \times (1 / \text{dual}\,\mathbf{x}) = \mathbf{x} / (\text{dual}\,\mathbf{x}) = [1, 1].$$

The Kaucher addition and multiplication are linked by the following distributivity relations. Whatever are $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{KR}$,

- conditional distributivity—see [20]:

$$\mathbf{x} \times \mathbf{y} + (\text{imp}\,\mathbf{x}) \times \mathbf{z} \ \subseteq\ \mathbf{x} \times (\mathbf{y} + \mathbf{z}) \ \subseteq\ \mathbf{x} \times \mathbf{y} + (\text{pro}\,\mathbf{x}) \times \mathbf{z}.$$

Its following three special cases are also useful:

- subdistributivity: if $\mathbf{x} \in \mathbb{IR}$ then $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$;
- superdistributivity: if $\mathbf{x} \in \overline{\mathbb{IR}}$ then $\mathbf{x} \times (\mathbf{y} + \mathbf{z}) \supseteq \mathbf{x} \times \mathbf{y} + \mathbf{x} \times \mathbf{z}$;
- distributivity: if $x \in \mathbb{R}$ then $x \times (\mathbf{y} + \mathbf{z}) = x \times \mathbf{y} + x \times \mathbf{z}$.

Some other distributive laws can be found in [13]. Another useful property of the Kaucher arithmetic is its monotonicity with respect to the inclusion: whatever are $\circ \in \{+, \times, -, \div\}$ and $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathbb{KR}$,

$$\mathbf{x} \subseteq \mathbf{x}' \wedge \mathbf{y} \subseteq \mathbf{y}' \implies (\mathbf{x} \circ \mathbf{y}) \subseteq (\mathbf{x}' \circ \mathbf{y}').$$

Finally, generalized interval vectors $\mathbf{x} \in \mathbb{KR}^n$ and generalized interval matrices $\mathbf{A} \in \mathbb{KR}^{n \times n}$ are defined like in the classical interval theory. The operations mid, rad, $|\,.\,|$, dual, pro, and imp are performed on vectors and matrices elementwise. It is easy to check that a diagonal interval matrix whose proper projections of the diagonal entries do not contain 0 is invertible, i.e. $0 \notin \mathbf{D}_{ii}$ for all $i \in [1..n]$ implies $\mathbf{D}^{-1}\mathbf{D} = \mathbf{D}\mathbf{D}^{-1} = I$ where $\mathbf{D}^{-1} = \text{diag}\left(\frac{1}{\text{dual}\,\mathbf{D}_{11}}, \ldots, \frac{1}{\text{dual}\,\mathbf{D}_{nn}}\right)$.

2.2. AE-Solution Sets

AE-solution sets generalize the united solution set of an interval system allowing arbitrary quantifications of the parameters, with the constraint that the universal quantifiers precede the existential ones inside the selecting predicate—this constraint being the source of the denomination A(ll)E(xist). They are formally defined in the following way:

DEFINITION 2.1 (Shary [16]). Consider an interval matrix $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{IR}^{n \times n}$, an interval vector $\mathbf{b} = (\mathbf{b}_k) \in \mathbb{IR}^n$ and a quantification $\alpha_{ij} \in \{\forall, \exists\}$ for the parameters a_{ij} and $\beta_k \in \{\forall, \exists\}$ for the parameters b_k. Then, the AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ is the following subset of $\mathbb{R}^n$:

$$\begin{aligned}\{x \in \mathbb{R}^n \ \mid\ & (\forall a_{\gamma_1} \in \mathbf{a}_{\gamma_1}) \ldots (\forall a_{\gamma_p} \in \mathbf{a}_{\gamma_p})(\forall b_{\delta_1} \in \mathbf{b}_{\delta_1}) \ldots (\forall b_{\delta_s} \in \mathbf{b}_{\delta_s}) \\ & (\exists a_{\bar{\gamma}_1} \in \mathbf{a}_{\bar{\gamma}_1}) \ldots (\exists a_{\bar{\gamma}_q} \in \mathbf{a}_{\bar{\gamma}_q}), (\exists b_{\bar{\delta}_1} \in \mathbf{b}_{\bar{\delta}_1}) \ldots (\exists b_{\bar{\delta}_t} \in \mathbf{b}_{\bar{\delta}_t}) \\ & (Ax = b)\},\end{aligned}$$

where

$$\begin{array}{rll} (i,j) \in \{\gamma_k \mid k \in [1\,..\,p]\} & \text{if and only if} & \alpha_{ij} = \forall, \\ (i,j) \in \{\overline{\gamma}_k \mid k \in [1\,..\,q]\} & \text{if and only if} & \alpha_{ij} = \exists, \\ k \in \{\delta_k \mid k \in [1\,..\,s]\} & \text{if and only if} & \beta_k = \forall, \\ k \in \{\overline{\delta}_k \mid k \in [1\,..\,t]\} & \text{if and only if} & \beta_k = \exists. \end{array}$$

The characteristic matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and characteristic vector $\mathbf{b} \in \mathbb{KR}^n$ of an AE-solution set $\Xi_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ are defined in the following way:

$$(\text{pro}\,\mathbf{A}) = \mathbf{A} \ \text{ and } \ \begin{cases} \alpha_{ij} = \forall \implies \mathbf{a}_{ij} \in \mathbb{IR}, \\ \alpha_{ij} = \exists \implies \mathbf{a}_{ij} \in \overline{\mathbb{IR}} \end{cases}$$

and

$$(\text{pro}\,\mathbf{b}) = \mathbf{b} \ \text{ and } \ \begin{cases} \beta_k = \forall \implies \mathbf{b}_{ij} \in \overline{\mathbb{IR}}, \\ \beta_k = \exists \implies \mathbf{b}_{ij} \in \mathbb{IR}. \end{cases}$$

Therefore, an AE-solution set is determined providing its characteristic matrix and vector. As a consequence, the notation $\Xi(\mathbf{A}, \mathbf{b})$ can be used instead of the notation $\Xi_{\alpha\beta}(\text{pro}\,\mathbf{A}, \text{pro}\,\mathbf{b})$.

Notation (Shary [16]). The AE-solution set which corresponds to the characteristic interval matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and to the characteristic vector $\mathbf{b} \in \mathbb{KR}^n$ is denoted by $\Xi(\mathbf{A}, \mathbf{b})$.

The usual united, tolerable and controllable solution sets are then described in the following way: given a proper interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and a proper interval vector $\mathbf{b} \in \mathbb{IR}^n$,

- united solution set: $\Sigma(\mathbf{A}, \mathbf{B}) = \Xi_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \Xi(\text{dual}\,\mathbf{A}, \mathbf{b})$;
- tolerable solution set: $\Xi_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \Xi(\mathbf{A}, \mathbf{b})$;
- controllable solution set: $\Xi_{\exists\forall}(\mathbf{A}, \mathbf{b}) = \Xi(\text{dual}\,\mathbf{A}, \text{dual}\,\mathbf{b})$.

One can notice that the conventions relating the proper/improper quality of a generalized interval to the quantification of the corresponding parameter are different for $\mathbf{A}$ and $\mathbf{b}$. These two conventions were chosen in this way because generalized intervals are not only a convenient model for the type of uncertainty but also an important analysis tool for the estimation of AE-solution sets. In particular, the conventions have been chosen so as to get the following central characterization theorem.

THEOREM 2.1 (Shary [16]). *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$. *Then,*

$$x \in \Xi(\mathbf{A}, \mathbf{b}) \iff \mathbf{A}x \subseteq \mathbf{b}.$$

This characterization theorem is used to link the AE-solution sets to some auxiliary interval equations in $\mathbb{KR}^n$.

2.3. Auxiliary Interval Equations Related to AE-Solution Sets

Three auxiliary interval equations in $\mathbb{KR}^n$ are now presented together with their relationship with the AE-solution sets. The auxiliary interval equation (2.1) is dedicated to inner estimation of AE-solution sets. A simple proof of the following theorem is given, which enlightens the importance of Theorem 2.1.

THEOREM 2.2 (Shary [16], Sainz [14]). *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set. Then any proper solution* $\mathbf{x} \in \mathbb{IR}^n$ *of the interval equation*

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{2.1}$$

is an inner estimate of $\Xi(\mathbf{A}, \mathbf{b})$, *i.e.* $\mathbf{x} \subseteq \Xi(\mathbf{A}, \mathbf{b})$.

Proof (taken from [16]). Let a proper interval vector $\mathbf{x}$ be a formal solution of the equation (2.1) and $\tilde{x} \in \mathbf{x}$. Then, in view of the monotonicity of the Kaucher arithmetic, we have

$$\mathbf{A}\tilde{x} \subseteq \mathbf{A}\mathbf{x} = \mathbf{b},$$

that is, $x \in \Xi(\mathbf{A}, \mathbf{b})$ by Theorem 2.1. □

Remark 2.1. The structure of modal intervals $(I^*(R), \subseteq, +, \times)$ and the one of generalized intervals $(\mathbb{KR}, \subseteq, +, \times)$ being isomorphic, a simplified presentation is done by translating the modal interval theorems into their equivalent formulation in the structure of generalized intervals.

Remark 2.2. The equation (2.1) is proved to have a unique solution provided that (pro $\mathbf{A}$) is strictly diagonally dominant [8], [14], [16].* See Section 3 for an improved uniqueness sufficient condition.

The next theorem presents the auxiliary interval equation (2.2) which is dedicated to outer estimation of AE-solution sets.

THEOREM 2.3 (Shary [16]). *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set. Suppose that***

$$\rho(|I - \text{dual}\,\mathbf{A}|) < 1.$$

* The condition for the existence of a unique solution provided by Markov [8] is more restrictive. However, the proof which is proposed in [8] can be modified so as to get the better condition stated in [14].

** The spectral radius $\rho(M)$ of a real matrix M is the maximum module of its eigenvalues. It will not have to be explicitly computed in the present paper.

Then, the following interval equation has a unique solution $\mathbf{x} \in \mathbb{KR}^n$*:*

$$\mathbf{x} = (I - \operatorname{dual} \mathbf{A})\mathbf{x} + \mathbf{b}. \tag{2.2}$$

Furthermore, if the solution $\mathbf{x}$ *is proper then it contains* $\Xi(\mathbf{A}, \mathbf{b})$*; if the solution* $\mathbf{x}$ *is not proper then* $\Xi(\mathbf{A}, \mathbf{b})$ *is empty.*

Remark 2.3. The previous theorem corresponds to Theorem 7.3 in [16]. However, its statement has been slightly modified to this equivalent formulation so as to be easier to apply.

Remark 2.4. In the special case of united solution set, i.e. $\mathbf{A} \in \overline{\mathbb{IR}}^{n\times n}$, a more general condition is provided in [9] for the interpretation of the solutions of the auxiliary interval equation $\mathbf{x} = (I - \operatorname{dual} \mathbf{A})\mathbf{x} + \mathbf{b}$ as an outer estimate.

Finally the auxiliary interval equation (2.3) is a preconditioned version of the second. The following inclusion has been proved in [16]: given $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{b} \in \mathbb{KR}^n$ and a nonsingular real matrix $C \in \mathbb{R}^{n\times n}$,

$$\Xi(\mathbf{A}, \mathbf{b}) \subseteq \Xi(C\mathbf{A}, C\mathbf{b}).$$

In contrast with the "naive left-preconditioning" presented in the introduction, the preconditioning matrix C is now applied directly to the characteristic matrix and vector. As the preconditioned AE-solution set $\Xi(C\mathbf{A}, C\mathbf{b})$ contains the original one, any outer estimate of preconditioned AE-solution set is also an outer estimate of the original one. In the spirit of the formal-algebraic approach, this preconditioning process is transfered into the auxiliary interval equation (2.2) so as to construct the auxiliary interval equation (2.3). Applying Theorem 2.3 to the left-preconditioned AE-solution set $\Xi(C\mathbf{A}, C\mathbf{b})$, the following theorem is raised:

THEOREM 2.4 (Shary [16]). *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set and* $C \in \mathbb{R}^{n\times n}$ *be a nonsingular real matrix. Suppose that*

$$\rho\big(|I - \operatorname{dual}(C\mathbf{A})|\big) < 1.$$

Then, the following interval equation has a unique solution $\mathbf{x} \in \mathbb{KR}^n$*:*

$$\mathbf{x} = \big(I - \operatorname{dual}(C\mathbf{A})\big)\mathbf{x} + C\mathbf{b}. \tag{2.3}$$

Furthermore, if the solution $\mathbf{x}$ *is proper then it contains* $\Xi(\mathbf{A}, \mathbf{b})$*; if the solution* $\mathbf{x}$ *is not proper then* $\Xi(\mathbf{A}, \mathbf{b})$ *is empty.*

Remark 2.5. The previous theorem corresponds to Theorem 7.3 in [16]. Its statement has been slightly modified, similarly to the statement of Theorem 2.3.

The introduction of strongly regular interval matrices allows to describe an important class of AE-solution sets which can be outer estimated using Theorem 2.4 together with the midpoint inverse preconditioning matrix, i.e. $C = (\operatorname{mid} \mathbf{A})^{-1}$. The strongly regular interval matrices are defined in the following way:

DEFINITION 2.2 (Neumaier [10]). Let $\mathbf{A} \in \mathbb{IR}^{n\times n}$ be an interval matrix. $\mathbf{A}$ is strongly regular if (mid $\mathbf{A}$) is a regular real matrix and the interval matrix $(\operatorname{mid}\mathbf{A})^{-1}\mathbf{A}$ is a regular interval matrix, i.e. contains only regular real matrices.

Finally, it is proved in [16] that if (pro $\mathbf{A}$) is strongly regular, the condition $\rho(|I - (\operatorname{mid}\mathbf{A})^{-1}\mathbf{A}|) < 1$ is satisfied, and Theorem 2.4 can be used with $C = (\operatorname{mid}\mathbf{A})^{-1}$ in order to compute an outer estimate of $\Xi(\mathbf{A}, \mathbf{b})$.

One More Auxiliary Interval Equation

Another auxiliary interval equation dedicated to outer estimation of AE-solution sets is provided in [15]. However, it is not in the scope of this paper to investigate the preconditioning of this last interval auxiliary equation. Given a characteristic matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and a characteristic vector $\mathbf{b} \in \mathbb{KR}^{n}$, the following system of interval equations is proposed:

$$\bigwedge_{i\in[1..n]} \mathbf{A}_{ii}\mathbf{x}_i + \sum_{\substack{j\in[1..n]\\ j\neq i}} \mathbf{A}_{ij}(\operatorname{dual}\mathbf{x}_j) = \mathbf{b}_i. \tag{2.4}$$

It is proved in [15] that, if this interval equation has a unique solution which is proper, this solution is an outer estimate of $\Xi(\mathbf{A}, \mathbf{b})$.

2.4. Algorithms Dedicated to the Resolution of the Auxiliary Interval Equations

Three different classes of algorithms have been proposed to solve the auxiliary interval equations presented in the previous subsection.

2.4.1. *Stationary Single Step Iterations*

First of all, the auxiliary interval equation to be solved is formally rewritten to an equivalent fixed point form $\mathbf{x} = F(\mathbf{x})$ using the rules of the Kaucher arithmetic:

- The auxiliary interval equation (2.1) is changed to its fixed point formulation in the following way: the i-th line of the auxiliary interval equation (2.1)

  $$\sum_{j\in[1..n]} \mathbf{A}_{ij}\mathbf{x}_j = \mathbf{b}_i$$

  is changed to

  $$\mathbf{x}_i = \frac{1}{\operatorname{dual}\mathbf{A}_{ii}}\left(\mathbf{b}_i - \sum_{\substack{j\in[1..n]\\ j\neq i}} \operatorname{dual}(\mathbf{A}_{ij}\mathbf{x}_j)\right).$$

  This transformation is possible provided that $0 \notin (\operatorname{pro}\mathbf{A}_{ii})$ for all $i \in [1..n]$. As the operations used to transform the system are group operations, the auxiliary interval equation (2.1) and its fixed point formulation are equivalent. This latter can be written in the following matrix form:

$$\mathbf{x} = \mathbf{D}^{-1}\Big(\mathbf{b} - \operatorname{dual}\big((\mathbf{L} + \mathbf{U})\mathbf{x}\big)\Big),$$

where $\mathbf{D}$, $\mathbf{L}$, and $\mathbf{U}$ are respectively the diagonal, lower, and upper parts of the interval matrix $\mathbf{A}$.

- The auxiliary interval equations (2.2) and (2.3) are already in a fixed point form.

Remark 2.6. Putting the auxiliary interval equation (2.4) into a fixed point form, one obtains the following equation:

$$\bigwedge_{i\in[1..n]} \mathbf{x}_i = \frac{1}{\operatorname{dual}\mathbf{A}_{ii}}\left(\mathbf{b}_i - \sum_{\substack{j\in[1..n]\\ j\neq i}} (\operatorname{dual}\mathbf{A}_{ij})\mathbf{x}_j\right).$$

This transformation is possible provided that $0 \notin (\operatorname{pro}\mathbf{A}_{ii})$ for all $i \in [1..n]$. It can be noticed that this fixed point form of the auxiliary interval equation (2.4) coincides with the generalized interval Gauss-Seidel method proposed in [16] where the intersection with the current outer estimate would not be computed, i.e. with equation (7.12) of [16].

Once the auxiliary interval equations are in a fixed-point form $\mathbf{x} = F(\mathbf{x})$, the stationary single-step method consists in iterating $\mathbf{x}^{(k+1)} = F(\mathbf{x}^{(k)})$, obtaining a sequence of intervals $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$ whose limit, if it exists, satisfies $\mathbf{x} = F(\mathbf{x})$ and hence also satisfies the initial auxiliary interval equation.

The next theorem gives a sufficient condition for the convergence of the stationary single-step method applied to the auxiliary interval equation (2.1).

THEOREM 2.5 (Sainz [14], Markov [3]). *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set. Provided that* $(\operatorname{pro}\mathbf{A})$ *is strictly diagonally dominant, the sequence* $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$, *defined by*

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\Big(\mathbf{b} - \operatorname{dual}\big((\mathbf{L} + \mathbf{U})\mathbf{x}^{(k)}\big)\Big),$$

where $\mathbf{D}$, $\mathbf{L}$, *and* $\mathbf{U}$ *are respectively the diagonal, lower, and upper parts of the interval matrix* $\mathbf{A}$, *converges to the unique solution of the interval equation* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *whatever is the initial interval vector* $\mathbf{x}^{(0)} \in \mathbb{KR}^n$.

The convergence sufficient condition provided by the previous theorem is very restrictive: Matrices which are not strictly diagonally dominant are usually met in applications.

The next theorem concerns the auxiliary interval equations (2.2) and (2.3)—when it is applied with $\mathbf{M} = (I - \operatorname{dual}(C\mathbf{A}))$ and $\mathbf{b} = C\mathbf{b}$.

THEOREM 2.6 (Shary [16]). *Let* $\mathbf{M} \in \mathbb{KR}^{n\times n}$ *be such that* $\rho(|\mathbf{M}|) < 1$ *and* $\mathbf{d} \in \mathbb{KR}^n$*. Then the sequence* $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$*, defined by*

$$\mathbf{x}^{(k+1)} = \mathbf{M}\mathbf{x}^{(k)} + \mathbf{d},$$

converges to the unique solution of the interval equation $\mathbf{x} = \mathbf{M}\mathbf{x} + \mathbf{d}$ *whatever is the initial interval vector* $\mathbf{x}^{(0)} \in \mathbb{KR}^n$*.*

Remark 2.7. The formulation of the latter theorem is not explicitly stated in [16] but corresponds to Theorem 7.2 of [16].

When using the midpoint inverse preconditioning matrix, the stationary single-step method applied to the auxiliary interval equation (2.3) converges provided that (pro $\mathbf{A}$) is strongly regular.

Finally, the convergence of the stationary single-step methods relies on contracting map theorems. As a consequence, the distance between $\mathbf{x}^{(k)}$ and the formal solution $\mathbf{x}^{(\infty)}$ is multiplied by the contracting factor at each iteration. The contracting factor being less than one, the limit up to machine precision is reached in a reasonable number of steps.

2.4.2. *Immersion in the Linear Space* $\mathbb{R}^{2n}$

This way to solve the auxiliary interval equations was introduced in [17], [18]. It consists in changing the auxiliary interval equation of $\mathbb{KR}^n$ into an auxiliary real equation of $\mathbb{R}^{2n}$. The following bijection from $\mathbb{KR}^n$ into $\mathbb{R}^{2n}$, called the "standard immersion" in [17], [18], has been introduced to this end:

$$\sigma : \mathbb{KR}^n \longrightarrow \mathbb{R}^{2n};\ \mathbf{x} \longmapsto (-\underline{\mathbf{x}_1}, \ldots, -\underline{\mathbf{x}_n}, \overline{\mathbf{x}_1}, \ldots, \overline{\mathbf{x}_n}).$$

One important property of this map is

$$\sigma(\mathbf{x}) = 0_{\mathbb{R}^{2n}} \iff \mathbf{x} = 0_{\mathbb{KR}^n},$$

where $0_{\mathbb{KR}^n}$ and $0_{\mathbb{R}^{2n}}$ respectively stand for the zero of $\mathbb{KR}^n$ and the zero of $\mathbb{R}^{2n}$. Thanks to this property, the map σ allows to solve an equation $\mathbf{F}(\mathbf{x}) = 0_{\mathbb{KR}^n}$ in $\mathbb{KR}^n$ by solving the auxiliary equation $\sigma(\mathbf{F}(\sigma^{-1}(x)) = 0_{\mathbb{R}^{2n}}$ in $\mathbb{R}^{2n}$. This transformation is then applied to the auxiliary interval equations (2.1) and (2.2)* after having put them in the form $\mathbf{F}(\mathbf{x}) = 0_{\mathbb{KR}^n}$.

A specific subdifferential Newton method has been proposed in [17], [18] which computes the solution of the equation $\sigma(\mathbf{F}(\sigma^{-1}(x)) = 0_{\mathbb{R}^{2n}}$ in $\mathbb{R}^{2n}$. The convergence of the subdifferential Newton method, when used to compute a solution of the auxiliary interval equation (2.2), has been studied in [17]. However, the obtained convergence sufficient condition is difficult to use in practice. Experimentations illustrate that when the single-step method converges, the subdifferential Newton

* It should also be applied to the auxiliary interval equations (2.3) and (2.4) but this has not been investigated.

method also converges. The subdifferential Newton method has two advantages in comparison with the stationary single-step methods:

- The subdifferential Newton method converges very quickly to the solution: in most examples, the solution of the auxiliary interval equation—up to machine precision—is reached in a number of iterations that does not exceed the dimension of the problem.
- The subdifferential Newton method converges in more cases than the stationary single-step methods—see [17], [18], and Example 6.3 of the present paper.

All the same, this does not disqualify the stationary single-step method: on one hand, the stationary single-step method is simpler to understand and to implement. On the other hand, the convergence rate of the stationary single-step method is reasonable, and even comparable to the one of the subdifferential Newton method when the contracting factor of the involved fixed point equation is small.

Remark 2.8. The subdifferential Newton method is considered in the present paper in its non-modified version. A specific preconditioning has been introduced in [17] in the context of outer estimation of united solution sets using the auxiliary interval equation (2.2). This preconditioning is somewhat similar to a left-preconditioning and its introduction into the framework of AE-solution sets is not in the scope of the present paper. Further informations about this specific preconditioning and the convergence of the subdifferential Newton method can be found in [11].

2.4.3. *Transformation of the Auxiliary Interval Equation into an Optimization Problem*

Two specific transformations have been proposed in [14] which change the auxiliary interval equations (2.1) and (2.4) into an optimization problem whose minimizer turns out to be the solution of the auxiliary interval equation. No precise study of the convergence and no comparison with the previous techniques have been conducted. It is not in the scope of the present paper to make such a comparison.

2.5. ON BEHAVIOR OF THE AUXILIARY INTERVAL EQUATION $\mathbf{Ax} = \mathbf{b}$

In the context of inner estimation of AE-solution sets, three reasons can make the resolution of the auxiliary interval equation $\mathbf{Ax} = \mathbf{b}$ inefficient:

- the equation has a solution which is not proper, and which therefore cannot be interpreted as an inner estimate of the united solution set;
- the equation has no solution;
- the equation has a manifold of solutions. In this case, the previously presented algorithms should not be able to compute one of the solutions.

Remark 2.9. The transformation of the resolution of the interval equation into an optimization problem may overcome the problem of the existence of a manifold of solutions by adding some additional constraints—maximization of the surface of the estimate for example. However, up to the knowledge of the author, that has not been investigated.

These three situations are met even on simple examples. The next example illustrates the first situation.

EXAMPLE 2.1. Consider $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} \in \mathbb{IR}^2$ and the corresponding united solution set

$$\Xi(A, \mathbf{b}) = \{x \in \mathbb{R}^2 \mid \exists b_1 \in \mathbf{b}_1,\ \exists b_2 \in \mathbf{b}_2,\ Ax = b\}.$$

An inner estimate of this united solution set may be obtained by solving the auxiliary interval equation $A\mathbf{x} = \mathbf{b}$. In this case, the matrix is triangular and $0 \neq A_{ii}$, hence the solution can be easily computed in the following way:

$$\begin{aligned} \mathbf{x}_2 &= \mathbf{b}_2, \\ \mathbf{x}_1 &= 0.5(\mathbf{b}_1 - 3\,\mathrm{dual}\,\mathbf{x}_2) = 0.5(\mathbf{b}_1 - 3\,\mathrm{dual}\,\mathbf{b}_2). \end{aligned}$$

Therefore, $\mathbf{x}$ is proper if and only if $\mathrm{rad}\,\mathbf{b}_1 \geq 3\,\mathrm{rad}\,\mathbf{b}_2$. In this case the resolution of the equation $A\mathbf{x} = \mathbf{b}$ leads to an inner estimate of the united solution set. However, if $\mathrm{rad}\,\mathbf{b}_1 < 3\,\mathrm{rad}\,\mathbf{b}_2$, the solution is not proper and hence is of no use for inner estimation. In this situation, the “squeezing of parameters” was proposed in [16]. It consists in studying an auxiliary AE-solution set which is included inside the original one by changing $\mathbf{b}$ to $\mathbf{b}' \subseteq \mathbf{b}$. For example, if $\mathbf{b} = ([-2, 2], [-10, 10])^T$, and hence the interval equation $A\mathbf{x} = \mathbf{b}$ has no proper solution, one can consider $\mathbf{b}' = ([-2, 2], [k, k+1])^T$ with $k \in [-10..9]$. Therefore, $\mathrm{rad}\,\mathbf{b}'_1 \geq 3\,\mathrm{rad}\,\mathbf{b}'_2$ and each equation $A\mathbf{x} = \mathbf{b}'$ with $k \in [-10..9]$ has a proper solution $\mathbf{x}$ which is a subset of the original united solution set. The united solution set can eventually be described by the union of 20 interval vectors.

The next example illustrates the last two situations.

EXAMPLE 2.2. Consider $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathbf{b} \in \mathbb{IR}^2$ and the corresponding united solution set

$$\Xi(A, \mathbf{b}) = \{x \in \mathbb{R}^2 \mid \exists b_1 \in \mathbf{b}_1,\ \exists b_2 \in \mathbf{b}_2,\ Ax = b\}.$$

An inner estimate of this united solution set may be obtained by solving the auxiliary interval equation $A\mathbf{x} = \mathbf{b}$. However, all the previous presented techniques diverge when applied so as to compute a solution of this interval equation. Following the idea proposed in [18], one can immerse this interval equation into $\mathbb{R}^4$ using the standard immersion σ presented in the previous subsection, obtaining the following linear equation of $\mathbb{R}^4$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\overline{\mathbf{b}_1} \\ -\overline{\mathbf{b}_2} \\ \underline{\mathbf{b}_1} \\ \underline{\mathbf{b}_2} \end{pmatrix},$$

where $\mathbf{x}_1 = [u, v]$ and $\mathbf{x}_2 = [y, z]$. The matrix of this linear equation is singular—check that $L4 = L1 + L3 - L2$. Therefore, two situations arise:

- The right hand side vector is compatible with this dependence, that is $\operatorname{rad}\mathbf{b}_1 = \operatorname{rad}\mathbf{b}_2$. In this case, there is a manifold of solutions to $A\mathbf{x} = \mathbf{b}$. When $\mathbf{b}_1 = \mathbf{b}_2 = [-1, 1]$, these solutions can be described by the following parametrization—personal communication of S. P. Shary:

  $$\mathbf{x}(t) = \frac{1}{2}\begin{pmatrix} [t-1,\ 1-t] \\ [-t-1,\ t+1] \end{pmatrix}.$$

  Within this manifold of solutions, the ones which satisfy $-1 \le t \le 1$ are proper, and hence are inner estimates of the united solution set.
- The right hand side vector is not compatible with the line dependence of the matrix A, i.e. $\operatorname{rad}\mathbf{b}_1 \ne \operatorname{rad}\mathbf{b}_2$. In this case, the equation $A\mathbf{x} = \mathbf{b}$ has no solution. Therefore, the right-hand side interval vector has to be squeezed before obtaining several manifolds of solutions.

In view of these two examples, the squeezing of parameters happens to be necessary so as to compute an inner estimate from the resolution of the auxiliary interval equation (2.1). However, this technique is difficult to use in more complicated situations. The preconditioned auxiliary interval equation dedicated to inner estimation to be proposed in the sequel will have a better behavior in many cases, and hence will simplify the construction of inner estimates.

3. An Improved Uniqueness Sufficient Condition for the Solution of the Auxiliary Interval Equation $\mathbf{Ax} = \mathbf{b}$

This section presents an improved uniqueness condition for the solution of the interval equation $\mathbf{Ax} = \mathbf{b}$. It is obtained through an improved convergence condition for the stationary single-step method applied to the auxiliary interval equation $\mathbf{Ax} = \mathbf{b}$. First of all, basic definitions related to interval H-matrices are recalled:

DEFINITION 3.1 (Neumaier [10]). Let $\mathbf{x} \in \mathbb{IR}$, $\mathbf{A} \in \mathbb{IR}^{n\times n}$, and $M \in \mathbb{R}^{n\times n}$.

1. $|\mathbf{x}| = \max\{|\underline{\mathbf{x}}|, |\overline{\mathbf{x}}|\}$ is the magnitude of $\mathbf{x}$;
2. $\langle\mathbf{x}\rangle = \min\{|\underline{\mathbf{x}}|, |\overline{\mathbf{x}}|\}$ if $0 \notin \mathbf{x}$ and $\langle\mathbf{x}\rangle = 0$ otherwise, is the mignitude of $\mathbf{x}$;
3. $\langle\mathbf{A}\rangle \in \mathbb{R}^{n\times n}$ is the comparison matrix of $\mathbf{A}$ and is defined by $\langle\mathbf{A}\rangle_{ii} = \langle\mathbf{A}_{ii}\rangle$ for the diagonal entries and $\langle\mathbf{A}\rangle_{ij} = -|\mathbf{A}_{ij}|$ if $i \ne j$;

4. M is a real M-matrix if and only if its off-diagonal entries are non-positive and there exists a real vector $u > 0$ such that $Mu > 0$;
5. $\mathbf{A}$ is an interval H-matrix if and only if its comparison matrix is a M-matrix;
6. $\mathbf{A}$ is strictly diagonally dominant if and only $\langle \mathbf{A} \rangle e > 0$, where $e = (1, \ldots, 1)^T$.

Strictly diagonally dominant interval matrices are interval H-matrices—see [10]. The following characterization of interval H-matrices is then proposed. Given $s \in \mathbb{R}^n$, the diagonal matrix whose diagonal entries are s_i is denoted by $\mathrm{diag}(s)$.

PROPOSITION 3.1. *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$. *Then,*

1. *for any* $s \in \mathbb{R}^n$ *such that* $s > 0$, $\langle \mathbf{A}\,\mathrm{diag}(s) \rangle = \langle \mathbf{A} \rangle\,\mathrm{diag}(s)$;
2. $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *is a H-matrix if and only if there exists* $s \in \mathbb{R}^n$, $s > 0$, *such that* $\mathbf{A}\,\mathrm{diag}(s)$ *is strictly diagonally dominant.*

Proof.
(1) The computation of the two matrix products is simple because $\mathrm{diag}(s)$ is diagonal. It remains to apply Proposition 1.6.1 from [10] in the following way: $\langle \mathbf{A}_{ii} s_i \rangle = \langle \mathbf{A}_{ii} \rangle |s_i| = \langle \mathbf{A}_{ii} \rangle s_i$ and $|\mathbf{A}_{ij} s_j| = |\mathbf{A}_{ij}||s_j| = |\mathbf{A}_{ij}| s_j$, where $|s_k| = s_k$ because $s_k > 0$.

(2) First notice that the off-diagonal entries of $\langle \mathbf{A} \rangle$ are non-positive by definition. Therefore, by the definition of an interval M-matrix, $\mathbf{A}$ is an interval H-matrix if and only if $\exists s > 0$, $\langle \mathbf{A} \rangle s > 0$. The vector s can be written as the product $\mathrm{diag}(s)e$ where $e = (1, \ldots, 1)^T$. Therefore, $\mathbf{A}$ is an interval H-matrix if and only if $\exists s > 0$, $\langle \mathbf{A} \rangle\,\mathrm{diag}(s)e > 0$. Applying the first case of the proposition, $\mathbf{A}$ is an interval H-matrix if and only if $\exists s > 0$, $\langle \mathbf{A}\,\mathrm{diag}(s) \rangle e > 0$, i.e. if and only if $\exists s > 0$ such that $\mathbf{A}\,\mathrm{diag}(s)$ is strictly diagonally dominant. □

This condition is not very convenient for checking if an interval matrix is a H-matrix. However, it will be useful for proving Theorem 3.1. The following lemma will also be needed.

LEMMA 3.1. *Let* $s \in \mathbb{R}^n$ *such that* $s > 0$, $\mathbf{M} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{z} \in \mathbb{KR}^n$. *The diagonal real matrix* $\mathrm{diag}(s)$ *is denoted by S. Then,* $(\mathbf{M}S)\mathbf{z} = \mathbf{M}(S\mathbf{z})$ *and* $(S\mathbf{M})\mathbf{z} = S(\mathbf{M}\mathbf{z})$. *This implies in particular that the continuous map* $\mathbf{x} \longmapsto S\mathbf{x}$ *is injective.*

Proof. Just notice that S being diagonal, the matrix products are highly simplified and no distribution has to be performed between + and ×. Now, the map $\mathbf{x} \longmapsto S\mathbf{x}$ is continuous as proved in [18] and is injective because $S\mathbf{x} = S\mathbf{y} \Longrightarrow S^{-1}(S\mathbf{x}) = S^{-1}(S\mathbf{x})$ which eventually implies $\mathbf{x} = \mathbf{y}$ thanks to the previously proved property. □

The next theorem improves the convergence condition proposed by Theorem 2.5 for the stationary single-step method applied to the auxiliary interval equation (2.1).

THEOREM 3.1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$. *Provided that* $(\mathrm{pro}\,\mathbf{A})$ *is a H-matrix, the sequence* $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$, *defined by*

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\Big(\mathbf{b} - \mathrm{dual}\big((\mathbf{L}+\mathbf{U})\mathbf{x}^{(k)}\big)\Big),$$

where $\mathbf{D}$, $\mathbf{L}$, *and* $\mathbf{U}$ *are respectively the diagonal, lower, and upper parts of the interval matrix* $\mathbf{A}$, *converges to the unique solution of the interval equation* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *whatever is the initial interval vector* $\mathbf{x}^{(0)} \in \mathbb{KR}^n$.

Proof. The interval matrix $(\mathrm{pro}\,\mathbf{A})$ being a H-matrix, the application of the proposition 3.1 proves that there exists $s > 0$ such that $(\mathrm{pro}\,\mathbf{A})S$ is strictly diagonally dominant, where $S := \mathrm{diag}(s)$. Define $\mathbf{A}' := \mathbf{A}S$—which is equivalent to $\mathbf{A}'S^{-1} = (\mathbf{A}S)S^{-1} = \mathbf{A}$ by Lemma 3.1. Thanks to the simplicity of the product with a diagonal matrix, it is trivial to check that $(\mathrm{pro}\,\mathbf{A}') = (\mathrm{pro}\,\mathbf{A})S$. As a consequence, $(\mathrm{pro}\,\mathbf{A}')$ is strictly diagonally dominant. Therefore, one can use Theorem 2.5 and proves that the sequence $(\mathbf{y}^{(k)})_{k\in\mathbb{N}}$, defined by

$$\mathbf{y}^{(k+1)} = \mathbf{D}'^{-1}\Big(\mathbf{b} - \mathrm{dual}\big((\mathbf{L}'+\mathbf{U}')\mathbf{y}^{(k)}\big)\Big),$$

converges to the unique solution of $\mathbf{A}'\mathbf{y} = \mathbf{b}$ whatever is $\mathbf{y}^{(0)}$—where $\mathbf{D}'$, $\mathbf{L}'$, and $\mathbf{U}'$ are respectively the diagonal, lower, and upper parts of the matrix $\mathbf{A}'$. So, choose $\mathbf{y}^{(0)} = S^{-1}\mathbf{x}^{(0)}$. We now prove by induction that $\mathbf{y}^{(k)} = S^{-1}\mathbf{x}^{(k)}$ for all $k \in \mathbb{N}$. Suppose that $\mathbf{y}^{(k)} = S^{-1}\mathbf{x}^{(k)}$. So, by the definition of $\mathbf{y}^{(k+1)}$ we have

$$\mathbf{y}^{(k+1)} = \mathbf{D}'^{-1}\Big(\mathbf{b} - \mathrm{dual}\big((\mathbf{L}'+\mathbf{U}')(S^{-1}\mathbf{x}^{(k)})\big)\Big).$$

Using Lemma 3.1 we obtain

$$\mathbf{y}^{(k+1)} = \mathbf{D}'^{-1}\Big(\mathbf{b} - \mathrm{dual}\Big(\big((\mathbf{L}'+\mathbf{U}')S^{-1}\big)\mathbf{x}^{(k)}\Big)\Big).$$

Then, because of the simplicity of the product with a diagonal matrix, it is trivial to get from $\mathbf{A}'S^{-1} = \mathbf{A}$ that $(\mathbf{L}'+\mathbf{U}')S^{-1} = (\mathbf{L}+\mathbf{U})$ and that $\mathbf{D}'S^{-1} = \mathbf{D}$, this latter implying $\mathbf{D}'^{-1} = S^{-1}\mathbf{D}^{-1}$. Therefore, we obtain

$$\mathbf{y}^{(k+1)} = (S^{-1}\mathbf{D}^{-1})\Big(\mathbf{b} - \mathrm{dual}\big((\mathbf{L}+\mathbf{U})\mathbf{x}^{(k)}\big)\Big).$$

We apply once more Lemma 3.1 and obtain

$$\mathbf{y}^{(k+1)} = S^{-1}\Big(\mathbf{D}^{-1}\Big(\mathbf{b} - \mathrm{dual}\big((\mathbf{L}+\mathbf{U})\mathbf{x}^{(k)}\big)\Big)\Big),$$

which eventually implies $\mathbf{y}^{(k+1)} = S^{-1}\mathbf{x}^{(k+1)}$ thanks to the inductive definition of $\mathbf{x}^{(k+1)}$. We have therefore proved that $\mathbf{y}^{(k)} = S^{-1}\mathbf{x}^{(k)}$ for all $k \in \mathbb{N}$. Now left multiply both terms of the previous equality by S and apply Lemma 3.1 to obtain $S\mathbf{y}^{(k)} = S(S^{-1}\mathbf{x}^{(k)}) = \mathbf{x}^{(k)}$, which holds for all $k \in \mathbb{N}$. The map S being continuous, the sequence $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$ is the image of the convergent sequence $(\mathbf{y}^{(k)})_{k\in\mathbb{N}}$, which implies that it also converges. Furthermore, the limits $\mathbf{x}^{(\infty)}$ and $\mathbf{y}^{(\infty)}$ also satisfy $\mathbf{y}^{(\infty)} = S^{-1}\mathbf{x}^{(\infty)}$. Now, as $\mathbf{A}'\mathbf{y}^{(\infty)} = \mathbf{b}$, we have $\mathbf{A}'(S^{-1}\mathbf{x}^{(\infty)}) = \mathbf{b}$, that is, by definition of $\mathbf{A}'$, $(\mathbf{A}S)(S^{-1}\mathbf{x}^{(\infty)}) = \mathbf{b}$. This eventually implies $\mathbf{A}\mathbf{x}^{(\infty)} = \mathbf{b}$ thanks to Lemma 3.1. Up to now, we have proved that the sequence $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$ defined in statement of the theorem converges to a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. It hence remains to prove that this solution is the *unique* solution of this equation. We suppose that $\tilde{\mathbf{x}} \neq \mathbf{x}^{(\infty)}$ also satisfies $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}$ and prove that this implies a contradiction. Then consider $\tilde{\mathbf{y}} := S^{-1}\tilde{\mathbf{x}}$ which is different of $\mathbf{y}^{(\infty)} = S^{-1}\mathbf{x}^{(\infty)}$ because the map S^{-1} is injective by Lemma 3.1. However, $\mathbf{A}'\tilde{\mathbf{y}} = (\mathbf{A}S)(S^{-1})\tilde{\mathbf{x}} = \mathbf{A}\tilde{\mathbf{x}}$ which is eventually equal to $\mathbf{b}$ by supposition. Therefore, $\mathbf{y}^{(\infty)}$ and $\tilde{\mathbf{y}}$ are two different solutions of $\mathbf{A}'\mathbf{y} = \mathbf{b}$, which eventually contradicts the uniqueness of the solution of this equation entailed by Theorem 2.5. □

The next example illustrates the improved convergence condition provided by the previous theorem.

EXAMPLE 3.1. Let $\mathbf{A} = \begin{pmatrix} 150, 1 \\ [-50, 50], 1 \end{pmatrix} \in \mathbb{IR}^{2\times 2}$ and $\mathbf{b} = \begin{pmatrix} [10, 20] \\ [-10, -20] \end{pmatrix} \in \mathbb{KR}^2$. The interval matrix $\mathbf{A}$ is a H-matrix—use Proposition 3.1 with $s = (0.01, 1)^T$—but not diagonally dominant. The application of the iteration process of Theorem 3.1 leads to a sequence which converges to $\mathbf{x}^{(\infty)} = [0.0, 0.4] \times [10, -40]$. One can check that $\mathbf{A}\mathbf{x}^{(\infty)} = \mathbf{b}$.

Remark 3.1. The subdifferential Newton method also converges to the unique solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ of the previous example.

The next corollary of Theorem 3.1 is an improved version of Theorem 6.7 in [16].

COROLLARY 3.1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *be such that* (pro $\mathbf{A}$) *is a H-matrix and* $\mathbf{b} \in \mathbb{KR}^n$, *then the equation* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a unique solution.*

Proof. This corollary is a light version of Theorem 3.1. □

4. Right-Preconditioned Auxiliary Interval Equations Dedicated to AE-Solution Sets Estimation

Given a characteristic matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$, a characteristic vector $\mathbf{b} \in \mathbb{KR}^n$, a nonsingular matrix $C \in \mathbb{R}^{n\times n}$ and a vector $\tilde{x} \in \mathbb{R}^n$, the relationships between the following two right-preconditioned auxiliary interval equations and the AE-solution set $\Xi(\mathbf{A}, \mathbf{b})$ are now investigated:

$$\big((\text{pro}\,\mathbf{A})C\big)\mathbf{u} = \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}, \tag{4.1}$$

$$\mathbf{u} = \big(I - (\text{pro}\,\mathbf{A})C\big)\mathbf{u} + \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}. \tag{4.2}$$

The natural choices $C = (\mathrm{mid}\,\mathbf{A})^{-1}$ and $\tilde{x} = (\mathrm{mid}\,\mathbf{A})^{-1}(\mathrm{mid}\,\mathbf{b})$ will be shown to have interesting consequences at the end of this section and in the next section. The following two propositions provide some intermediary results which will be useful for the interpretation of the solutions of these auxiliary interval equations.

PROPOSITION 4.1. *Given* $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{b} \in \mathbb{KR}^n$ *and a nonsingular real matrix* $C \in \mathbb{R}^{n\times n}$,

1. $\mathbf{A}$ *is proper implies* $\{Cu \mid u \in \Xi(\mathbf{A}C, \mathbf{b})\} \subseteq \Xi(\mathbf{A}, \mathbf{b})$;
2. $\mathbf{A}$ *is improper implies* $\Xi(\mathbf{A}, \mathbf{b}) \subseteq \{Cu \mid u \in \Xi(\mathbf{A}C, \mathbf{b})\}$.

Proof.
(1) Consider a vector $u \in \mathbb{R}^n$. Then, using the characterization Theorem 2.1, $u \in \Xi(\mathbf{A}C, \mathbf{b})$ implies $(\mathbf{A}C)u \subseteq \mathbf{b}$. Now, the matrix $\mathbf{A}$ being proper, the subdistributivity law implies $\mathbf{A}(Cu) \subseteq (\mathbf{A}C)u \subseteq \mathbf{b}$—consider C as a proper interval matrix and u as a proper interval vector. Finally, $\mathbf{A}(Cu) \subseteq \mathbf{b}$ implies $Cu \in \Xi(\mathbf{A}, \mathbf{b})$ thanks to the characterization Theorem 2.1. Therefore, we have proved that $u \in \Xi(\mathbf{A}C, \mathbf{b})$ implies $Cu \in \Xi(\mathbf{A}, \mathbf{b})$, which is another formulation of statement to be proved.

(2) The proof of the second statement is conducted in a similar way: Using the superdistributivity law—and considering C as an improper interval matrix and u as an improper interval vector—one can check that $Cu \in \Xi(\mathbf{A}, \mathbf{b})$ implies $u \in \Xi(\mathbf{A}C, \mathbf{b})$. □

The following lemma will be used to prove Proposition 4.2.

LEMMA 4.1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{a} \in \mathbb{KR}$. *Then, for any* $x \in \mathbb{R}$ *and* $\tilde{x} \in \mathbb{R}$,

1. $(\mathrm{imp}\,\mathbf{a})(x - \tilde{x}) + \mathbf{a}\tilde{x} \subseteq \mathbf{a}x \subseteq (\mathrm{pro}\,\mathbf{a})(x - \tilde{x}) + \mathbf{a}\tilde{x}$;
2. $(\mathrm{imp}\,\mathbf{A})(x - \tilde{x}) + \mathbf{A}\tilde{x} \subseteq \mathbf{A}x \subseteq (\mathrm{pro}\,\mathbf{A})(x - \tilde{x}) + \mathbf{A}\tilde{x}$.

Proof.
(1) The conditional distributivity law applied to the product $\mathbf{a}((x - \tilde{x}) + \tilde{x})$ raises the following inclusions:

$$(\mathrm{imp}\,\mathbf{a})(x - \tilde{x}) + \mathbf{a}\tilde{x} \subseteq \mathbf{a}\big((x - \tilde{x}) + \tilde{x}\big) \subseteq (\mathrm{pro}\,\mathbf{a})(x - \tilde{x}) + \mathbf{a}\tilde{x}.$$

It just remains to notice that $\mathbf{a}((x - \tilde{x}) + \tilde{x}) = \mathbf{a}x$.

(2) Thanks to case (1), one concludes that $\forall i \in [1..n]$ and $\forall j \in [1..n]$,

$$(\mathrm{imp}\,\mathbf{A}_{ij})(x_j - \tilde{x}_j) + \mathbf{A}_{ij}\tilde{x}_j \subseteq \mathbf{A}_{ij}x_j \subseteq (\mathrm{pro}\,\mathbf{A}_{ij})(x_j - \tilde{x}_j) + \mathbf{A}_{ij}\tilde{x}_j.$$

Then, using the inclusion monotonicity of the addition, one can sum this inclusions for $j \in [1..n]$. Therefore, $\forall i \in [1..n]$,

$$\sum_{j\in[1..n]} (\mathrm{imp}\,\mathbf{A}_{ij})(x_j - \tilde{x}_j) + \sum_{j\in[1..n]} \mathbf{A}_{ij}\tilde{x}_j \subseteq \sum_{j\in[1..n]} \mathbf{A}_{ij}x_j$$
$$\subseteq \sum_{j\in[1..n]} (\mathrm{pro}\,\mathbf{A}_{ij})(x_j - \tilde{x}_j) + \sum_{j\in[1..n]} \mathbf{A}_{ij}\tilde{x}_j,$$

which eventually corresponds to the statement to be proved. □

Then comes the last intermediary result. The translated of a set $\mathbb{E} \subseteq \mathbb{R}^n$ by vector $\tilde{x} \in \mathbb{R}^n$ is denoted by $\tilde{x} + \mathbb{E}$.

PROPOSITION 4.2. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{b} \in \mathbb{KR}^n$ *and* $\tilde{x} \in \mathbb{R}^n$. *Then,*

$$\tilde{x} + \Xi\big(\text{pro}\,\mathbf{A}, \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}\big) \subseteq \Xi(\mathbf{A}, \mathbf{b}) \subseteq \tilde{x} + \Xi\big(\text{imp}\,\mathbf{A}, \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}\big).$$

Proof. Consider the first inclusion and any $x \in \tilde{x} + \Xi(\text{pro}\,\mathbf{A}, \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x})$. Notice that $x \in \tilde{x} + \Xi(\text{pro}\,\mathbf{A}, \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x})$ is equivalent to $(x - \tilde{x}) \in \Xi(\text{pro}\,\mathbf{A}, \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x})$. The characterization Theorem 2.1 proves that this implies $(\text{pro}\,\mathbf{A})(x - \tilde{x}) \subseteq \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}$. Then, adding $\mathbf{A}\tilde{x}$ to both sides of the inclusion and noticing that $\mathbf{A}\tilde{x} - (\text{dual}\,\mathbf{A})\tilde{x} = 0$, one obtains $(\text{pro}\,\mathbf{A})(x - \tilde{x}) + \mathbf{A}\tilde{x} \subseteq \mathbf{b}$. Finally, Lemma 4.1 implies $\mathbf{A}x \subseteq \mathbf{b}$ which means $x \in \Xi(\mathbf{A}, \mathbf{b})$ by the characterization Theorem 2.1. The second inclusion is proved in a similar way. □

The relationships between the auxiliary interval equations (4.1) and (4.2) and the AE-solution set $\Xi(\mathbf{A}, \mathbf{b})$ are now presented through Theorems 4.1 and 4.2. The next theorem proves that the resolution of the equation (4.1) allows the construction of a skew box included inside the related AE-solution set.

THEOREM 4.1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set,* $C \in \mathbb{R}^{n\times n}$ *be a nonsingular matrix and* $\tilde{x} \in \mathbb{R}^n$. *Consider any proper solution* $\mathbf{u} \in \mathbb{IR}^n$ *of the equation*

$$\big((\text{pro}\,\mathbf{A})C\big)\mathbf{u} = \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}.$$

Then, the skew box $\{\tilde{x} + Cu \mid u \in \mathbf{u}\}$ *is included in* $\Xi(\mathbf{A}, \mathbf{b})$.

Proof. As $\mathbf{u}$ is a proper solution of the equation $\mathbf{A}'\mathbf{u} = \mathbf{b}'$ where $\mathbf{A}' := (\text{pro}\,\mathbf{A})C$ and $\mathbf{b}' := \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}$, Theorem 2.2 proves the following inclusion:

$$\mathbf{u} \subseteq \Xi(\mathbf{A}', \mathbf{b}').$$

Then, $\mathbf{A}'$ being proper, Proposition 4.1 can be used so as to prove that

$$\{Cu \mid u \in \Xi(\mathbf{A}', \mathbf{b}')\} \subseteq \Xi(\text{pro}\,\mathbf{A}, \mathbf{b}').$$

Thanks to $\mathbf{u} \subseteq \Xi(\mathbf{A}', \mathbf{b}')$, the previous inclusion entails the following one:

$$\{Cu \mid u \in \mathbf{u}\} \subseteq \Xi(\text{pro}\,\mathbf{A}, \mathbf{b}').$$

Then, translating both previous subsets of $\mathbb{R}^n$ by the vector $\tilde{x}$ preserves the previous inclusion, and we obtain the following inclusion:

$$\{\tilde{x} + Cu \mid u \in \mathbf{u}\} \subseteq \tilde{x} + \Xi(\text{pro}\,\mathbf{A}, \mathbf{b}').$$

Finally, Proposition 4.2 proves that $\tilde{x} + \Xi(\text{pro}\,\mathbf{A}, \mathbf{b}') \subseteq \Xi(\mathbf{A}, \mathbf{b})$, which concludes the proof. □

The next theorem proves that the resolution of the equation (4.2) allows the construction of a skew box which contains the related AE-solution set.

THEOREM 4.2. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^{n}$ *be the characteristics of an AE-solution set,* $C \in \mathbb{R}^{n\times n}$ *be a nonsingular real matrix and* $\tilde{x} \in \mathbb{R}^n$*. Suppose that*

$$\rho\big(|I - (\mathrm{pro}\,\mathbf{A})C|\big) < 1.$$

Then, the following equation has a unique solution $\mathbf{u} \in \mathbb{KR}^n$*:*

$$\mathbf{u} = \big(I - (\mathrm{pro}\,\mathbf{A})C\big)\mathbf{u} + \mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}.$$

Furthermore, if the solution $\mathbf{u}$ *is proper then the skew box* $\{\tilde{x} + Cu \mid u \in \mathbf{u}\}$ *contains* $\Xi(\mathbf{A}, \mathbf{b})$*; if the solution* $\mathbf{u}$ *is not proper then* $\Xi(\mathbf{A}, \mathbf{b})$ *is empty.*

Proof. First notice that $\mathrm{dual}((\mathrm{imp}\,\mathbf{A})C) = (\mathrm{pro}\,\mathbf{A})C$. So, the interval equation of the statement of the theorem can be written $\mathbf{u} = (I - \mathrm{dual}\,\mathbf{A}')\mathbf{x} + \mathbf{b}'$, where $\mathbf{A}' = (\mathrm{imp}\,\mathbf{A})C$ and $\mathbf{b}' = \mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}$. Therefore, by Theorem 2.3, it has an unique solution $\mathbf{u}$. Suppose that $\mathbf{u}$ is proper. As $\mathbf{u}$ is the solution of the equation $\mathbf{u} = (I - \mathrm{dual}\,\mathbf{A}')\mathbf{x} + \mathbf{b}'$, Theorem 2.3 proves the following inclusion:

$$\Xi(\mathbf{A}', \mathbf{b}') \subseteq \mathbf{u}.$$

Then, $\mathbf{A}'$ being improper, Proposition 4.1 can be used so as to prove that

$$\Xi(\mathrm{imp}\,\mathbf{A}, \mathbf{b}') \subseteq \{Cu \mid u \in \Xi(\mathbf{A}', \mathbf{b}')\}.$$

Thanks to $\Xi(\mathbf{A}', \mathbf{b}') \subseteq \mathbf{u}$, the previous inclusion entails the following one:

$$\Xi(\mathrm{imp}\,\mathbf{A}, \mathbf{b}') \subseteq \{Cu \mid u \in \mathbf{u}\}.$$

Then, translating both previous subsets of $\mathbb{R}^n$ by the vector $\tilde{x}$ preserves the previous inclusion, and we obtain the following inclusion:

$$\tilde{x} + \Xi(\mathrm{imp}\,\mathbf{A}, \mathbf{b}') \subseteq \{\tilde{x} + Cu \mid u \in \mathbf{u}\}.$$

Finally, Proposition 4.2 proves that $\Xi(\mathbf{A}, \mathbf{b}) \subseteq \tilde{x} + \Xi(\mathrm{imp}\,\mathbf{A}, \mathbf{b}')$, which concludes this part of the proof. Now, suppose that $\mathbf{u}$ is not proper. Then, Theorem 2.3 proves that $\Xi(\mathbf{A}', \mathbf{b}')$ is empty. Therefore, using Proposition 4.1, $\Xi(\mathrm{imp}\,\mathbf{A}, \mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x})$ is also empty—as it is contained inside $\{Cu \mid u \in \emptyset\} = \emptyset$. Finally, using Proposition 4.2, one concludes that $\Xi(\mathbf{A}, \mathbf{b})$ is empty. □

The use of the midpoint inverse preconditioning matrix together with the introduction of strongly regular interval matrices allows the description of a class of AE-solution sets which can be estimated using Theorems 4.1 and 4.2. However, the strongly regular matrices have been introduced in the context of left-preconditioning, and their usefulness in the context of right-preconditioning first has to be established. This is the aim of the next proposition which is similar to

Proposition 4.1.1 in [10] but dedicated to right-preconditioning, i.e. involving the product $\mathbf{A}(\text{mid}\,\mathbf{A})^{-1}$ instead of $(\text{mid}\,\mathbf{A})^{-1}\mathbf{A}$.

PROPOSITION 4.3. *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *and suppose that* mid **A** *is regular. Then the following conditions are equivalent:*

1. **A** *is strongly regular;*
2. $\mathbf{A}^T$ *is strongly regular;*
3. $\mathbf{A}(\text{mid}\,\mathbf{A})^{-1}$ *is regular;*
4. $\rho((\text{rad}\,\mathbf{A})|(\text{mid}\,\mathbf{A})^{-1}|) < 1$*;*
5. $||I - \mathbf{A}(\text{mid}\,\mathbf{A})^{-1}||_v < 1$ *for some* $v > 0$*;*
6. $\mathbf{A}(\text{mid}\,\mathbf{A})^{-1}$ *is a H-matrix.*

Proof.
($1 \Leftrightarrow 2$): Proposition 4.1.1 from [10].

($2 \Leftrightarrow 3$): Thanks to the first case, **A** is strongly regular if and only if $\mathbf{A}^T$ is strongly regular. By definition of strongly regular matrices, this is true if and only if $(\text{mid}\,\mathbf{A}^T)^{-1}\mathbf{A}^T$ is regular. Therefore, as $(\text{mid}\,\mathbf{A}^T)^{-1} = ((\text{mid}\,\mathbf{A})^{-1})^T$, this is equivalent to the statement that $((\text{mid}\,\mathbf{A})^{-1})^T\mathbf{A}^T$ is regular. An interval matrix is regular if and only if its transpose is regular, therefore **A** is strongly regular if and only if $(((\text{mid}\,\mathbf{A})^{-1})^T\mathbf{A}^T)^T$ is regular. This is finally equivalent to $\mathbf{A}(\text{mid}\,\mathbf{A})^{-1}$ is regular.

($3 \Rightarrow 4$), ($4 \Rightarrow 5$), ($5 \Rightarrow 6$), and ($6 \Rightarrow 3$) are proved similarly to the proofs provided in [10], changing the definition of $\mathbf{B}_0$ to $\mathbf{B}_0 = \mathbf{A}\hat{\mathbf{A}}^{-1}$. □

As a consequence, when the characteristic matrix of an AE-solution set has a strongly regular proper projection, the unique solution of the midpoint inverse right-preconditioned auxiliary interval equations (4.1) and (4.2) can be computed. From now on, the preconditioning will be performed using $C = (\text{mid}\,\mathbf{A})^{-1}$ and $\tilde{x} = (\text{mid}\,\mathbf{A})^{-1}(\text{mid}\,\mathbf{b})$.

4.1. Inner Estimation

If pro **A** is strongly regular, Proposition 4.3 proves that

$$(\text{pro}\,\mathbf{A})(\text{mid}\,\mathbf{A})^{-1}$$

is a H-matrix. In this case, Theorem 3.1 proves that the stationary single-step method can compute the unique solution of the auxiliary interval equation (4.1). Therefore the problem of the existence of a manifold of solutions cannot arise with the right-preconditioned interval equation provided that the proper projection of the involved characteristic interval matrix is strongly regular. In the cases where the characteristic matrix **A** is thin, the right-preconditioned auxiliary interval equation (4.1) is simplified to the following one:

$$\mathbf{u} = \mathbf{b} - A\tilde{x}.$$

The solution $\mathbf{u}$ is then always proper because $A\tilde{x} = (\mathrm{mid}\,\mathbf{b}) \subseteq \mathbf{b}$ holds—that is $0 \subseteq \mathbf{b} - A\tilde{x}$ holds. Furthermore, the skew box inner estimate

$$\{\tilde{x} + A^{-1}u \mid u \in \mathbf{b} - \mathrm{mid}\,\mathbf{b}\} = \{A^{-1}u \mid u \in \mathbf{b}\}$$

is the exact solution set. However, the situation is more complicated when the matrix $\mathbf{A}$ is not thin, as illustrated in Section 6.

4.2. Outer Estimation

Thanks to Proposition 4.3, the condition

$$\rho\big(|I - (\mathrm{pro}\,\mathbf{A})(\mathrm{mid}\,\mathbf{A})^{-1}|\big) < 1$$

is satisfied provided $(\mathrm{pro}\,\mathbf{A})$ is strongly regular. Therefore, Theorem 2.6 proves that the stationary single-step method computes the unique solution of right-preconditioned auxiliary interval equation (4.2), which eventually leads to a—possibly empty—skew box that contains $\Xi(\mathbf{A}, \mathbf{b})$.

5. On the Quality of Skew Box Estimates

This section provides two theoretical results on the quality of the skew box estimates which are built using the solutions of the auxiliary interval equations (4.1) and (4.2) and using the midpoint inverse preconditioning. The l_∞-norm will be used for interval and real vectors and matrices:

$$||\mathbf{x}|| = \max_{i \in [1..n]} |\mathbf{x}_i| \qquad \text{and} \qquad ||\mathbf{M}|| = \max_{i \in [1..n]} \sum_{j \in [1..n]} |\mathbf{M}_{ij}|.$$

For any strictly positive real vector $v \in \mathbb{R}^n$, the scaled maximum norms are defined for real vectors and matrices in the following way:

$$||x||_v = \max_{i \in [1..n]} \frac{|x_i|}{v_i} \qquad \text{and} \qquad ||M||_v = \max_{i \in [1..n]} \sum_{j \in [1..n]} |M_{ij}| \, \frac{v_j}{v_i}.$$

Notice that $||x|| = ||x||_v$ for $v = (1, \ldots, 1)^T$ and that $||x|| \leq ||v|| \; ||x||_v$. The following Hausdorff distance between compact subsets E and E' of $\mathbb{R}^n$

$$\mathrm{dist}_H(E, E') = \max\Big\{ \max_{x \in E} \min_{x' \in E'} ||x - x'||, \max_{x' \in E'} \min_{x \in E} ||x - x'|| \Big\},$$

which is the usual distance between classical intervals vectors, is also considered as the distance between skew boxes.

5.1. Preliminary Results

The following lemma gives an upper bound on the Hausdorff distance between two skew boxes in the specific case which will be useful for the coming developments.

LEMMA 5.1. *Let* $M \in \mathbb{R}^{n\times n}$, $\tilde{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{IR}^n$, *and* $\mathbf{u}' \in \mathbb{IR}^n$ *such that* $\mathbf{u} \subseteq \mathbf{u}'$. *Denote* $\{\tilde{x} + Mu \mid u \in \mathbf{u}\}$ *and* $\{\tilde{x} + Mu \mid u \in \mathbf{u}'\}$ *respectively by* $\mathbf{s}$ *and* $\mathbf{s}'$. *Then,*

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') \leq ||M|| \text{dist}_H(\mathbf{u}, \mathbf{u}').$$

Proof. By definition of the Hausdorff distance,

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') = \max\left\{ \max_{x\in\mathbf{s}} \min_{x'\in\mathbf{s}'} ||x - x'||, \max_{x'\in\mathbf{s}'} \min_{x\in\mathbf{s}} ||x - x'|| \right\}.$$

As $\mathbf{s} \subseteq \mathbf{s}'$ because $\mathbf{u} \subseteq \mathbf{u}'$ by hypothesis, we have $\max_{x\in\mathbf{s}} \min_{x'\in\mathbf{s}'} ||x - x'|| = 0$ and hence

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') = \max_{x'\in\mathbf{s}'} \min_{x\in\mathbf{s}} ||x - x'||.$$

Using the definition of a skew box, this is written in the following way:

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') = \max_{u'\in\mathbf{u}'} \min_{u\in\mathbf{u}} ||Mu + \tilde{x} - (Mu' + \tilde{x})|| = \max_{u'\in\mathbf{u}'} \min_{u\in\mathbf{u}} ||M(u - u')||.$$

Now, as $||M(u - u')|| \leq ||M||\,||u - u'||$ and because performing the minimum or maximum on greater values gives a greater result, we obtain the following inequality:

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') \leq \max_{u'\in\mathbf{u}'} \min_{u\in\mathbf{u}} ||M||\,||u - u'||.$$

Finally, the statement of the theorem is obtained by factoring the positive number $||M||$ with respect to min and max operations. □

The following two lemmas about the midpoint and radius of interval matrices products are needed.

LEMMA 5.2. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$, $\mathbf{B} \in \mathbb{KR}^{n\times n}$. *Denote* $\text{mid}\,\mathbf{A}$ *and* $\text{mid}\,\mathbf{B}$ *respectively by* $\hat{\mathbf{A}}$ *and* $\hat{\mathbf{B}}$. *Then,*

- $\text{mid}(\mathbf{A} \pm \mathbf{B}) = \hat{\mathbf{A}} \pm \hat{\mathbf{B}}$,
- $\text{rad}(\mathbf{A} \pm \mathbf{B}) = (\text{rad}\,\mathbf{A}) + (\text{rad}\,\mathbf{B})$,
- $\text{mid}(\mathbf{AB}) = \hat{\mathbf{A}}\hat{\mathbf{B}}$ *if* $\mathbf{A}$ *is thin, or* $\mathbf{B}$ *is thin,*
- $\text{mid}(\mathbf{AB}) = 0$ *if* $\hat{\mathbf{A}} = 0$, *or* $\hat{\mathbf{B}} = 0$, *where* 0 *stands for the null matrix.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *are proper, then the following relations also hold:*

- $\text{rad}(\mathbf{AB}) = |\mathbf{A}|(\text{rad}\,\mathbf{B})$ *if* $\mathbf{A}$ *is thin or* $\hat{\mathbf{B}} = 0$,
- $\text{rad}(\mathbf{AB}) = (\text{rad}\,\mathbf{A})|\mathbf{B}|$ *if* $\mathbf{B}$ *is thin or* $\hat{\mathbf{A}} = 0$.

Proof. The first three relations are direct consequences of the expressions of the Kaucher arithmetic. Notice that the radius involved in the second relation may be negative. For example, $[-1, 1] + [5, 0] = [4, 1]$ and the radius rule is satisfied as $2+(-5) = -3$. The fourth holds because the product of any intervals is centered on 0 when one of them is centered on 0. The fifth is taken from [10, Proposition 3.1.12] and the sixth is proved in a similar way to the fifth. □

Remark 5.1. The properties stated by previous lemma also stand for addition of interval vectors, and multiplication of interval matrices by interval vectors.

LEMMA 5.3. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^n$ *be such that* $\operatorname{mid} \mathbf{A} = I$, $0 \notin (\operatorname{pro} \mathbf{A}_{ii})$ *for all* $i \in [1..n]$ *and* $\operatorname{mid} \mathbf{b} = 0$. *Then any solution* $\mathbf{x}$ *of the equation* $\mathbf{Ax} = \mathbf{b}$ *satisfies* $(\operatorname{mid} \mathbf{x}) = 0$.

Proof. As $0 \notin \mathbf{A}_{ii}$, the equation $\mathbf{Ax} = \mathbf{b}$ can be written in the following form:

$$\mathbf{x} = \mathbf{D}^{-1}\Big(\mathbf{b} - \operatorname{dual}\big((\mathbf{L} + \mathbf{U})\mathbf{x}\big)\Big).$$

Then, by Lemma 5.2, as $\mathbf{L}+\mathbf{U}$ is centered on 0, so is $(\mathbf{L}+\mathbf{U})\mathbf{x}$. Then, as $\mathbf{b}$ is centered on 0, $\mathbf{b} - \operatorname{dual}((\mathbf{L} + \mathbf{U})\mathbf{x})$ is also centered on 0 and finally $\mathbf{D}^{-1}(\mathbf{b} - \operatorname{dual}((\mathbf{L} + \mathbf{U})\mathbf{x}))$ is also centered on 0. □

Finally, the next lemma provides an upper bound for the distance between the respective solutions of two specific real linear equations which will be met in the sequel.

LEMMA 5.4. *Let* $\Delta \in \mathbb{R}^{n\times n}$ *be a nonnegative matrix,* $v \in \mathbb{R}^n$ *a strictly positive vector such that* $||\Delta||_v < 1$, *and* $\beta \in \mathbb{R}^n$. *If* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^n$ *satisfy respectively*

$$(I - \Delta)x = \beta \qquad \text{and} \qquad (I + \Delta)y = \beta$$

then

$$||x - y|| \;\leq\; 2\,||v||\,\frac{||\Delta||_v\;||\beta||_v}{1 - ||\Delta||_v^2}.$$

Proof. This lemma relies upon Neumann series, see [10]: If $||\Delta||_v < 1$—which implies $||-\Delta||_v < 1$—then both $I - \Delta$ and $I + \Delta$ are invertible and the following two equalities hold:

$$(I - \Delta)^{-1} = \sum_{k=0}^{\infty} \Delta^k \qquad \text{and} \qquad (I + \Delta)^{-1} = \sum_{k=0}^{\infty}(-1)^k\Delta^k.$$

Now, as Δ is nonnegative we have

$$\sum_{k=0}^{\infty} |(-1)^k\Delta^k| = \sum_{k=0}^{\infty} |\Delta^k| = \sum_{k=0}^{\infty} \Delta^k.$$

The last sum being convergent, both $\sum_{k=0}^{\infty} \Delta^k$ and $\sum_{k=0}^{\infty} (-1)^k \Delta^k$ are absolutely convergent. As a consequence, we can use associativity and commutativity so as to rearrange the terms of the infinite sum $(I-\Delta)^{-1}-(I+\Delta)^{-1}$ in the following way:

$$(I-\Delta)^{-1} - (I+\Delta)^{-1} = 2\Delta + 2\Delta^3 + 2\Delta^5 + \cdots = 2\Delta \sum_{k=0}^{\infty} \Delta^{2k}.$$

As by hypothesis $||\Delta||_v < 1$, the geometric sum $\sum_{k=0}^{\infty} ((||\Delta||_v)^2)^k$ is convergent and

$$\sum_{k=0}^{\infty} \left((||\Delta||_v)^2\right)^k = \frac{1}{1-(||\Delta||_v)^2}.$$

As a consequence, we have

$$||(I-\Delta)^{-1} - (I+\Delta)^{-1}||_v \leq 2||\Delta||_v \frac{1}{1-(||\Delta||_v)^2}.$$

Then, $x - y = ((I-\Delta)^{-1} - (I+\Delta)^{-1})\beta$ and the following inequality is raised:

$$||x-y||_v = ||\left((I-\Delta)^{-1} - (I+\Delta)^{-1}\right)\beta||_v \leq 2\,||\Delta||_v \frac{1}{1-(||\Delta||_v)^2}\,||\beta||_v.$$

Finally, thanks to $||x-y|| \leq ||v||\,||x-y||_v$, one concludes the proof. □

5.2. On the Distance between the Inner and Outer Skew Box Estimates

The next theorem provides a general result about the accuracy of the inner and outer skew box estimates built thanks to the solutions of the right-preconditioned auxiliary interval equations (4.1) and (4.2).

THEOREM 5.1. *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *be such that* pro $\mathbf{A}$ *is strongly regular and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of an AE-solution set. Consider the interval equations (4.1) and (4.2) with*

$$C = \hat{\mathbf{A}}^{-1} \qquad \textit{and} \qquad \tilde{x} = \hat{\mathbf{A}}^{-1}(\text{mid}\,\mathbf{b}),$$

where $\hat{\mathbf{A}}$ *stands for* mid $\mathbf{A}$. *Furthermore suppose that the respective solutions* $\mathbf{u}$ *and* $\mathbf{u}'$ *of these equations are proper. Then, for all strictly positive vector* $v \in \mathbb{R}^n$ *such that*[⋆]

$$||I - (\text{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}||_v < 1,$$

⋆ Such vectors v indeed exist because pro $\mathbf{A}$ is strongly regular.

the Hausdorff distance between the skew boxes inner estimate $\mathbf{s} := \{\tilde{x} + Cu \mid u \in \mathbf{u}\}$ *and outer estimate* $\mathbf{s}' := \{\tilde{x} + Cu \mid u \in \mathbf{u}'\}$ *is bounded in the following way:*

$$\mathrm{dist}_H(\mathbf{s}, \mathbf{s}') \leq 2\,||v||\,||\hat{\mathbf{A}}^{-1}||\,\frac{||I - (\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}||_v\,||\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}||_v}{1 - \big(||I - (\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}||_v\big)^2}.$$

Proof. The proper interval vectors $\mathbf{u}$ and $\mathbf{u}'$ satisfy respectively the interval equations (4.1) and (4.2), that is,

$$\begin{aligned}\big((\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big)\mathbf{u} &= \mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x},\\ \mathbf{u}' &= \big(I - (\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big)\mathbf{u}' + \mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}.\end{aligned}$$

As $\tilde{x} = \hat{\mathbf{A}}^{-1}(\mathrm{mid}\,\mathbf{b})$, one can use Lemma 5.2 and concludes that $\mathrm{mid}(\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}) = 0$. Furthermore notice that using Lemma 5.2 we have

$$\mathrm{mid}\big((\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big) = I \qquad \text{and} \qquad \mathrm{mid}\big(I - (\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big) = 0.$$

Therefore one finally comes to the conclusion that $\mathrm{mid}\,\mathbf{u} = 0$ by Lemma 5.3 and that $\mathrm{mid}\,\mathbf{u}' = 0$ by Lemma 5.2. Now notice that $\mathrm{rad}(\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}) = |\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}|$, because $\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}$ is centered on 0, and denote this real vector by β. Then, applying rad to the two interval equations and using Lemma 5.2—$\mathbf{u}$ and $\mathbf{u}'$ are proper and centered on 0—one gets the following relations:

$$\begin{aligned}|(\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{u}) &= \beta,\\ \mathrm{rad}\,\mathbf{u}' &= \Delta(\mathrm{rad}\,\mathbf{u}') + \beta,\end{aligned}$$

where $\Delta = |I - (\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}| = \mathrm{rad}((\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1})$ by Lemma 5.2. Now, by Proposition 3.1.10 from [10], we have

$$|(\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}| = I + \mathrm{rad}\big((\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big) = I + \Delta.$$

The situation is summarized by the following relations:

$$\begin{aligned}(I + \Delta)(\mathrm{rad}\,\mathbf{u}) &= \beta,\\ (I - \Delta)(\mathrm{rad}\,\mathbf{u}') &= \beta.\end{aligned}$$

Once these relations are established, for all strictly positive vector v such that $||\Delta||_v < 1$, we can use Lemma 5.4 hence obtaining the following bound on the distance between $(\mathrm{rad}\,\mathbf{u})$ and $(\mathrm{rad}\,\mathbf{u}')$:

$$||(\mathrm{rad}\,\mathbf{u}) - (\mathrm{rad}\,\mathbf{u}')|| \leq 2\,\frac{||\Delta||_v}{1 - ||\Delta||_v^2}\,||\beta||_v\,||v||.$$

As both $\mathbf{u}$ and $\mathbf{u}'$ are centered on 0, we have

$$\mathrm{dist}_H(\mathbf{u}, \mathbf{u}') = ||(\mathrm{rad}\,\mathbf{u}) - (\mathrm{rad}\,\mathbf{u}')||,$$

and the wanted inequality is eventually obtained using Lemma 5.1 which proves that $\text{dist}_H(\mathbf{s}, \mathbf{s}') \leq ||\hat{\mathbf{A}}^{-1}||\text{dist}_H(\mathbf{u}, \mathbf{u}')$. □

Remark 5.2. In practice, the previous theorem should be used with the l_∞-norm, i.e. $v = (1, \ldots, 1)^T$ and $||v|| = 1$. This requires $||I - (\text{pro}\,\mathbf{A})(\text{mid}\,\mathbf{A})^{-1}|| < 1$, that is, $(\text{pro}\,\mathbf{A})(\text{mid}\,\mathbf{A})^{-1}$ has to be strictly diagonally dominant, which is more restrictive than the condition "pro **A** is strongly regular," but is often satisfied.

Like in the case of the upper bound proposed in [12], if both

$$||I - (\text{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}|| \leq \varepsilon \qquad \text{and} \qquad ||\mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}|| \leq \varepsilon,$$

where ε can be chosen such that $\varepsilon \ll 1$ if the radius of pro **A** and pro **b** are small enough, the previous upper bound becomes

$$\text{dist}_H(\mathbf{s}, \mathbf{s}') \leq 2\,||\hat{\mathbf{A}}^{-1}||\,\frac{\varepsilon^2}{1-\varepsilon^2} \approx 2\,||\hat{\mathbf{A}}^{-1}||\,\varepsilon^2.$$

This proves that the skew box estimates can provide accurate estimates of the AE-solution set.

5.3. Comparison of Right and Left Preconditioning Processes in Some Special Cases of Outer Estimation

The next theorem proves that, in the context of outer estimation of AE-solution sets with improper characteristic matrix* and when the inverse mid-point preconditioning matrix is chosen, the right-preconditioning is more accurate than the left-preconditioning.

THEOREM 5.2. *Let* $\mathbf{A} \in \overline{\mathbb{IR}}^{n\times n}$ *such that* pro **A** *is strongly regular and* $\mathbf{b} \in \mathbb{KR}^n$ *be the characteristics of a nonempty AE-solution set. Consider the interval equations (2.3) and (4.2) with*

$$C = (\text{mid}\,\mathbf{A})^{-1} \qquad \textit{and} \qquad \tilde{x} = (\text{mid}\,\mathbf{A})^{-1}(\text{mid}\,\mathbf{b}).$$

Denote the unique proper solutions of these equations respectively by **x** *and* **u**. *Then,*

$$\square\{\tilde{x} + Cu \mid u \in \mathbf{u}\} = \mathbf{x},$$

where $\square\{\tilde{x} + Cu \mid u \in \mathbf{u}\}$ *stands for the interval hull of the skew box.*

Proof. First, notice that the solutions of the interval equations (2.3) and (4.2) are unique because pro **A** is strongly regular and are proper because the AE-solution set is supposed to be nonempty—use Theorems 2.4 and 4.2. Now, from the proof of Theorem 5.1, we know that mid **u** and rad **u** satisfy the following relations:

$$\begin{aligned} \text{mid}\,\mathbf{u} &= 0, \\ \text{rad}\,\mathbf{u} &= \text{rad}\left((\text{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\right)(\text{rad}\,\mathbf{u}) + \text{rad}\left(\mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}\right). \end{aligned}$$

* For example, a united solution set or a controllable solution set.

Notice that $(\mathrm{rad}\,\mathrm{pro}\,\mathbf{A}) = |\,\mathrm{rad}\,\mathbf{A}|$ and use Lemma 5.2 to prove that

$$\mathrm{rad}\big((\mathrm{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}\big) = |\,\mathrm{rad}\,\mathbf{A}|\;|\hat{\mathbf{A}}^{-1}|.$$

We now use the hypothesis that $\mathbf{A}$ is improper: This implies that $\mathrm{dual}\,\mathbf{A}$ is proper and hence we can use Lemma 5.2 so as to obtain

$$\mathrm{rad}\,\big(\mathbf{b} - (\mathrm{dual}\,\mathbf{A})\tilde{x}\big) = \mathrm{rad}\,\mathbf{b} + \big(\mathrm{rad}(\mathrm{pro}\,\mathbf{A})\big)|\tilde{x}| = \mathrm{rad}\,\mathbf{b} + |\,\mathrm{rad}\,\mathbf{A}|\;|\tilde{x}|.$$

As a consequence, $\mathrm{rad}\,\mathbf{u}$ satisfies

$$\mathrm{rad}\,\mathbf{u} = |\,\mathrm{rad}\,\mathbf{A}|\;|\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{u}) + \mathrm{rad}\,\mathbf{b} + |\,\mathrm{rad}\,\mathbf{A}|\;|\tilde{x}|.$$

Now left multiply the both side of this equality by $|\hat{\mathbf{A}}^{-1}|$ to obtain the following one:

$$|\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{u}) = |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|\;|\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{u}) + |\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{b}) + |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|\;|\tilde{x}|.$$

Denote $\square\{\tilde{x} + \hat{\mathbf{A}}^{-1}u \mid u \in \mathbf{u}\}$ by $\mathbf{y}$. It is easy to check that

$$\mathbf{y} = \tilde{x} + \hat{\mathbf{A}}^{-1}\mathbf{u}.$$

Therefore, $\mathrm{mid}\,\mathbf{y} = \tilde{x}$ and $\mathrm{rad}\,\mathbf{y} = |\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{u})$. Therefore we obtain the following equation for $\mathrm{rad}\,\mathbf{y}$:

$$\mathrm{rad}\,\mathbf{y} = |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|(\mathrm{rad}\,\mathbf{y}) + |\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{b}) + |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|\;|\tilde{x}|.$$

We now investigate the equation satisfied by $\mathrm{rad}\,\mathbf{x}$. One can check that applying the midpoint and radius rules of Lemma 5.2 to the interval equation (2.3) leads to the following relations—just notice that $|\mathbf{x}| = (\mathrm{rad}\,\mathbf{x}) + |\,\mathrm{mid}\,\mathbf{x}| = (\mathrm{rad}\,\mathbf{x}) + |\tilde{x}|$ on the way to get the equation for $\mathrm{rad}\,\mathbf{x}$:

$$\begin{aligned} \mathrm{mid}\,\mathbf{x} &= \tilde{x}, \\ \mathrm{rad}\,\mathbf{x} &= |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|(\mathrm{rad}\,\mathbf{x}) + |\hat{\mathbf{A}}^{-1}|(\mathrm{rad}\,\mathbf{b}) + |\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}|\;|\tilde{x}|. \end{aligned}$$

Therefore, $\mathrm{mid}\,\mathbf{y} = \mathrm{mid}\,\mathbf{x}$ and both $\mathrm{rad}\,\mathbf{y}$ and $\mathrm{rad}\,\mathbf{x}$ satisfy the same equation. Finally, as $|\hat{\mathbf{A}}^{-1}|\;|\,\mathrm{rad}\,\mathbf{A}| = |I - \hat{\mathbf{A}}^{-1}(\mathrm{pro}\,\mathbf{A})|$, this latter having a spectral radius strictly less than 1 because $\mathrm{pro}\,\mathbf{A}$ is strongly regular, the Schröder theorem proves that the equation shared by $\mathrm{rad}\,\mathbf{y}$ and $\mathrm{rad}\,\mathbf{x}$ has a unique solution. Finally, $\mathrm{mid}\,\mathbf{x} = \mathrm{mid}\,\mathbf{y}$ and $\mathrm{rad}\,\mathbf{x} = \mathrm{rad}\,\mathbf{y}$ which implies $\mathbf{x} = \mathbf{y}$. □

6. Experiments

The following auxiliary intervals equations are considered:

- the non-preconditioned interval equation (2.1) dedicated to inner estimation. Its solution will be denoted by $\mathbf{x}^I$ when available;

- the left-preconditioned interval equation (2.3), dedicated to outer estimation. Its solution will be denoted by $\mathbf{x}^O$ when available;
- the right-preconditioned interval equation (4.1), dedicated to inner estimation. Its solution will be denoted by $\mathbf{u}^I$ when available and the corresponding skew box estimate by $\mathbf{s}^I$;
- the right-preconditioned interval equation (4.2), dedicated to outer estimation. Its solution will be denoted by $\mathbf{u}^O$ when available and the corresponding skew box estimate by $\mathbf{s}^O$.

The auxiliary interval equations are preconditioned using

$$C = (\operatorname{mid}\mathbf{A})^{-1} \qquad \text{and} \qquad \tilde{x} = (\operatorname{mid}\mathbf{A})^{-1}(\operatorname{mid}\mathbf{b}).$$

The surface—hypervolume—of the skew box estimates are computed thanks to the classical formula $\operatorname{vol}\{Mu \mid u \in \mathbf{u}\} = (\det M)(\operatorname{vol}\mathbf{u})$—which generalizes the formula for the hypervolume of a box $\mathbf{u}$ obtained with $M = I$.

The next example illustrates that the skew box estimates can be more precise than the box estimates.

EXAMPLE 6.1. Consider the united solution set $\Xi(\operatorname{dual}\mathbf{A}, \mathbf{b})$ where

$$\mathbf{A} = \begin{pmatrix} 0.4490 \pm 8.432\text{E}^{-4}, & 0.4385 \pm 8.578\text{E}^{-4} \\ 0.4385 \pm 8.062\text{E}^{-4}, & 0.6510 \pm 6.098\text{E}^{-4} \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 5.657 \pm 1.029\text{E}^{-3} \\ 0.6119 \pm 1.583\text{E}^{-3} \end{pmatrix},$$

taken from [1], [2]. Each auxiliary interval equation has a unique proper solution:

$$\begin{aligned} \mathbf{x}^I &\approx ([0.9968, 1.0018], [0.2645, 0.2690])^T, \\ \mathbf{x}^O &\approx ([0.9740, 1.0246], [0.2458, 0.2877])^T, \\ \mathbf{u}^I &\approx ([-2.06\text{E}^{-3}, 2.06\text{E}^{-3}], [-2.51\text{E}^{-3}, 2.51\text{E}^{-3}])^T, \\ \mathbf{u}^O &\approx ([-2.13\text{E}^{-3}, 2.13\text{E}^{-3}], [-2.58\text{E}^{-3}, 2.58\text{E}^{-3}])^T. \end{aligned}$$

It can be noticed that $\mathbf{x}^I$ was reached after 134 iterations of the stationary single-step method and after only 1 iteration of the subdifferential Newton method. Thanks to the preconditioning, the contracting factor of the stationary single-step method is closer to 0 and the solution $\mathbf{u}^I$ was reached after only 6 iterations of the stationary single-step method.

The comparison of the following Hausdorff distances proves that the estimation by skew boxes is more efficient than the estimation by interval vectors:

$$\operatorname{dist}_H(\mathbf{x}^O, \mathbf{x}^I) \approx 2.3\text{E}^{-2} \qquad \text{and} \qquad \operatorname{dist}_H(\mathbf{s}^O, \mathbf{s}^I) \approx 7.9\text{E}^{-4}.$$

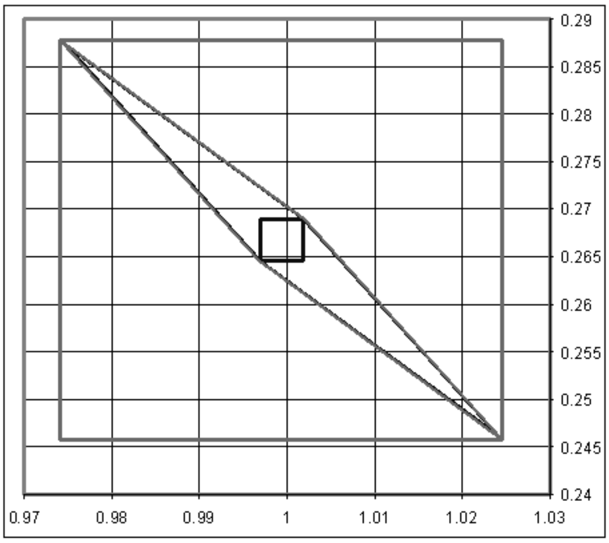

Figure 2. The united solution set of Example 6.1. Represented are the interval vector inner and outer estimates and the skew box inner and outer estimates—the two skew box estimates are too close to be discernible.

The upper bound for the Hausdorff distance provided by Theorem 5.1 is $9.4E^{-4}$ which is accurate. The interval hull of $\mathbf{s}^O$ is $\tilde{x} + (\mathrm{mid}\,\mathbf{A})^{-1}\mathbf{u}^O$, which is equal to $\mathbf{x}^O$. Theorem 5.2 is therefore confirmed. The two box estimates and the two skew box estimates are represented in the graphic of Figure 2. This graphic shows that the skew box estimates provide a much more accurate description than the box estimates.

The next example illustrates that the inner estimation through the resolution of the right-preconditioned interval equation can be less sensitive to perturbations of the right hand side interval vector than the inner estimation through the non-preconditioned interval equation.

EXAMPLE 6.2. Consider the united solution set of the previous example where $\mathbf{b}$ is changed to

$$\mathbf{b}' = \mathrm{mid}\,\mathbf{b} + \mathrm{diag}(d_1, d_2)(\mathbf{b} - \mathrm{mid}\,\mathbf{b}),$$

where $0 < q < 0.5$ and either $d_1 = 1$ and $d_2 = q$ or $d_1 = q$ and $d_2 = 1$—that is, the center of $\mathbf{b}$ is kept unchanged whereas the radius of one component of $\mathbf{b}$ is multiplied by q. In these cases, the solution of $(\mathrm{dual}\,\mathbf{A})\mathbf{x} = \mathbf{b}'$ is improper. This should be overcome squeezing the parameters.

The right-preconditioned interval equation is not sensitive to this alteration of Example 6.1: The solution remains proper, and the quality of the skew box estimates is similar to the previous example.

The next two examples illustrate that the scope of the formal-algebraic approach to inner estimation of AE-solution sets is indeed enlarged thanks to the right-preconditioned auxiliary interval equation (4.1).

EXAMPLE 6.3. Consider the tolerable solution set $\Xi(\mathbf{A},\mathbf{b})$ where

$$\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{a} & 0 & 0 & 0 & 0 \\ \mathbf{a} & \mathbf{a} & \mathbf{a} & 0 & 0 & 0 \\ 0 & \mathbf{a} & \mathbf{a} & \mathbf{a} & 0 & 0 \\ 0 & 0 & \mathbf{a} & \mathbf{a} & \mathbf{a} & 0 \\ 0 & 0 & 0 & \mathbf{a} & \mathbf{a} & \mathbf{a} \\ 0 & 0 & 0 & 0 & \mathbf{a} & \mathbf{a} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, 0.9] \end{pmatrix}.$$

and $\mathbf{a} = [0.999, 1.001]$. Looking for an inner estimate of $\Xi(\mathbf{A},\mathbf{b})$, one may consider the interval equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. When applied so as to compute a solution to this equation, the stationary single-step method diverges whereas the subdifferential Newton method converges. However, the solution provided by the subdifferential Newton method is not proper. The right thing to do would be to squeeze the parameters so as to obtain another interval equation $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ which would have a proper solution. However, the squeezing technique is difficult to apply because we have no information on the conditions which would provide a proper solution. The right-preconditioned interval equation $(\text{pro}\,\mathbf{A})C = \mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}$ has the following proper solution:

$$\begin{aligned} \mathbf{u}^I \approx\ & \big([-0.0981, 0.0981], [-0.0959, 0.0959], [-0.0941, 0.0941], \\ & [-0.0941, 0.0941], [-0.0959, 0.0959], [-0.0981, 0.0981]\big)^T. \end{aligned}$$

It is computed in 6 iterations of the stationary single-step method and 1 iteration of the subdifferential Newton method. Furthermore, the upper bound for the Hausdorff distance between the inner and outer skew box estimate is

$$2\,||\hat{\mathbf{A}}^{-1}||\,\frac{||I - (\text{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}||\ ||\mathbf{b} - (\text{dual}\,\mathbf{A})\tilde{x}||}{1 - \big(||I - (\text{pro}\,\mathbf{A})\hat{\mathbf{A}}^{-1}||\big)^2} \approx 1.5\text{E}^{-2}.$$

The hypervolume of the solution set is greater than the hypervolume of its inner estimate, which is

$$\text{vol}\,\mathbf{s}^I = \det(C)\,\text{vol}\,\mathbf{u}^I \approx (0.19)^6.$$

Therefore, the skew box inner estimate is accurate—if the solution set was an hypercube, its dimension would be greater than 0.19 which is much bigger than 1.5E^{-2}. Finally, the interval vector outer estimate $\mathbf{x}^O$ is this time strictly included inside the interval hull of the skew box outer estimate $\mathbf{s}^O$—which proves that Theorem 5.2 cannot be extended to tolerable solution sets. However, the hypervolume of the skew box outer estimate is approximately $(0.20)^6$ which is much smaller than the one of the box outer estimate which is $(0.65)^6$. Therefore, both outer estimates are complementary, even if the skew box outer estimate seems to give a more accurate description of the tolerable solution set.

EXAMPLE 6.4. Consider the tolerable solution set of the previous example where $\mathbf{A}_{16}$ is changed to -1. In this case, both the stationary single-step method and the subdifferential Newton method diverge when applied to $\mathbf{Ax} = \mathbf{b}$. This is certainly caused by the bad behavior of the interval equation $\mathbf{Ax} = \mathbf{b}$ presented in Section 2.5, i.e. the interval equation has a manifold of solutions or no solution at all. Building an inner estimate from the resolution of an equation like $\mathbf{Ax} = \mathbf{b}$ using the techniques presented in Section 2.5, i.e. squeezing parameters and building an explicit description of the manifold of solutions, seems to be an impossible job in this example.

The right-preconditioned interval equation is not sensitive to the alteration performed on Example 6.3. Its solution is

$$\begin{aligned}\mathbf{u}^I \approx \big(&[-0.0982, 0.0982], [-0.0960, 0.0960], [-0.0942, 0.0942],\\ &[-0.0943, 0.0943], [-0.0963, 0.0963], [-0.0985, 0.0985]\big)^T,\end{aligned}$$

which is proper. It is computed with the same number of iterations as previously. Furthermore, the quality of the skew box estimates is similar to the previous example.

The next example provides a situation where the right-preconditioned interval equation dedicated to inner estimation gives a worse result than the non-preconditioned interval equation.

EXAMPLE 6.5. Consider the united solution set $\Xi(\mathrm{dual}\,\mathbf{A}, \mathbf{b})$ where

$$\mathbf{A} = \begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} [-1,1] \\ [-10,10] \end{pmatrix}.$$

The interval equation $(\mathrm{dual}\,\mathbf{A})\mathbf{x} = \mathbf{b}$ has the following proper solution:

$$\mathbf{x}^I = \big([-0.5, 0.5], [-4.75, 4.75]\big)^T,$$

which is an inner estimate of the united solution set. The right-preconditioned interval equation has a solution which is not proper on this example.

Finally, an amazing behavior of the auxiliary interval equations dedicated to inner estimation is obtained modifying the previous example in the following way:

EXAMPLE 6.6. Consider the united solution set of the previous example where $\mathbf{A}$ is changed keeping its center and multiplying its radius by $q = 0.1$, i.e.

$$\mathbf{A} = \begin{pmatrix} [2.45, 2.55] & [0.45, 0.55] \\ [1.45, 1.55] & [2.45, 2.55] \end{pmatrix}.$$

This time, the interval equation (dual $\mathbf{A}$)$\mathbf{x} = \mathbf{b}$ has a non proper solution whereas the right-preconditioned interval equation has a proper solution. Choosing any coefficient $0 \leq q < 0.1$ does not change anymore the proper/improper quality of the corresponding solutions.

7. Conclusion

The main contribution of this paper is a right-preconditioning process dedicated to inner and outer estimation of AE-solution sets: Two right-preconditioned interval equations in the space of generalized intervals are proposed. Under some conditions, the solutions of these intervals equations can be used so as to build inner and outer estimates of AE-solution sets under the form of parallelepipeds.

The newly introduced right-preconditioned interval equation dedicated to inner estimation presents three advantages: First, the existence of a unique solution to this interval equation was proved provided that the proper projection of the characteristic matrix of the AE-solution set is strongly regular—a situation which is likely to be met. This is indeed an improvement in comparison with the behavior of the original non-preconditioned interval equation dedicated to inner estimation: This latter could have a manifold of solutions or no solution at all when strongly regular interval matrices are involved, which increases the difficulty of its resolution. Second, when the solution of the interval equation is not proper, the "squeezing and inflating of parameters" was proposed in [16] so as to obtain an inner estimate of the AE-solution set. In some situations, the proposed right-preconditioned interval equation can be used instead of the "squeezing of parameters": On one hand, the computed description of the inner estimate is more complete as the parameters have not been squeezed. On the other hand, the squeezing of parameters can be difficult to apply, and even almost impossible to apply for some AE-solution sets. Therefore the right-preconditioned interval equation seems to be the only way to build an inner estimate of such AE-solution sets. However, it happens on some tested examples that the right-preconditioned interval equation has worse behavior than the non-preconditioned one and a deeper study remains to be conducted. Third, thanks to the preconditioning, the stationary single-step method has a improved convergence rate: It was shown to be improved by a factor superior to 20 on some tested examples.

The right-preconditioned interval equation dedicated to outer estimation presents the same advantages as the one pointed out in [12] in the restricted context of outer estimation of united solution sets: The estimation of AE-solution sets by parallelepipeds can be more accurate than the estimation by interval vectors. It was proved in particular that the newly introduced right-preconditioning is indeed more accurate than the left-preconditioning in the cases of united and controllable solution sets.

7.1. FUTURE WORK

These results on the quality of the right-preconditioning have been obtained using the midpoint inverse preconditioning. This preconditioning has the advantage of giving to the skew box estimates the general shape of the AE-solution set to be estimated. However, it is well known that the midpoint inverse preconditioning becomes inefficient for high dimension problems, and the proposed right-preconditioning should be further investigated when used with other preconditioning matrices. Also, if skew boxes have been used already in [4] and [7] in the context of ordinary differential equation, the practical issues of estimation of AE-solution sets by skew boxes have to be investigated.

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