Modal intervals revisited: a mean-value extension to generalized intervals

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Abstract. The modal intervals theory deals with quantified propositions in AE-form, i.e. universal quantifiers precede existential ones, where variables are quantified over continuous domains and with equality constraints. It allows to manipulate such quantified propositions computing only with bounds of intervals. A simpler formulation of this theory is presented. Thanks to this new framework, a mean-value extension to generalized intervals (intervals whose bounds are not constrained to be ordered) is defined. Its application to the validation of quantified propositions is illustrated.

1 Introduction

Classical intervals are used in many situations to rigorously compute with interval domains instead of reals. This usually leads to outer approximations of sets defined by existentially quantified constraints (see [20, 13, 5]). A fundamental concept of the classical intervals theory is the extension of continuous functions to intervals. Such extensions allow to compute outer approximation of functions ranges over boxes[3]. Some widely used interval extensions are the natural extension and the mean-value extension. The former consists in replacing real operations by their interval counterparts in the expression of the function. The latter consists in linearizing the original function before bounding its range using the natural extension (see [4]).

Example 1. Let us consider the function $f(x) = x^2 - x$ and the interval $[2, 3]$. Its natural extension raises $f([2, 3]) = [2, 3]^2 - [2, 3] = [1, 7]$ while its mean-value extension raises $f(2.5) + f'([2, 3]) \times ([2, 3] - 2.5) = [0.75, 6.75]$. Both computed intervals contain the range of f over $[2, 3]$ which is $\{f(x) \mid x \in [2, 3]\} = [2, 6]$.

One main drawback of interval extensions is that they compute supersets of function ranges which can be very pessimistic. In general, the larger interval arguments, the bigger the overestimation of the range is. So, interval extensions

[3] Cartesian product of intervals.

are likely to be used with small enough interval arguments, this situation being usually reached during a bisection algorithm. In this situation, the mean-value extension has an important advantage on the natural extension as it is generally much more accurate when applied to small enough intervals: formally, the mean-value extension has a quadratic order of convergence while the natural extension has a linear order of convergence (see [4]).

Alternatively one can interpret interval extensions using quantified propositions. For example, the interpretation $\{f(x) \mid x \in [2,3]\} \subseteq [1,7]$ of Example 1 can be equivalently stated by the following quantified proposition:

$$\big(\forall x \in [2,3]\big)\big(\exists z \in [1,7]\big)\big(f(x) = z\big).$$

Starting from this latter interpretation of interval extensions, the modal intervals theory (see [11, 12]) succeeded in providing richer interpretations involving more general quantifications, these modal interpretations having promising applications (see [17–19]). As in the context of classical intervals, the pessimism related to the modal evaluation of a function is a central problem. Still a lot of work has to be conducted so as to deal with realistic situations (see [19]). A step in this direction would be a mean-value extension like linearization process for the modal evaluation of a function.

In this paper, we present a simpler formulation of the modal intervals theory using only generalized intervals (intervals whose bounds are not constrained to be ordered). This new formulation of the theory allows to define a mean-value extension to generalized intervals which is compatible with the richer interpretations of modal interval extensions (for a detailed presentation see [2, 1]). Some didactic examples are presented illustrating both the underlying mechanisms and the potential applications.

Notations Following [9], intervals are closed, bounded and nonempty, and interval objects are denoted by boldface letters. The set of intervals is denoted by $\mathbb{IR} := \{[a, b] \mid a \in \mathbb{R}, b \in \mathbb{R}, a \leq b\}$. The set of generalized intervals is denoted by $\mathbb{KR} := \{[a, b] \mid a \in \mathbb{R}, b \in \mathbb{R}\}$. Integral intervals are denoted by $[m..n]$. We will use the following conventions for vectors of reals, intervals and functions: sets of indices are ordered by the usual lexicographic order and are denoted by calligraphic letters. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a set of indices, the vector $(x_{e_1}, ..., x_{e_n})^T \in \mathbb{R}^n$ is denoted by $x_{\mathcal{E}}$. So that the vector $(x_1, \ldots, x_n)^T$ is denoted by $x_{[1..n]}$. If no confusion is possible, the usual notation x will be used in place of $x_{[1..n]}$.

2 Extensions to generalized intervals

In this section, we define generalized interval extensions whose interpretation generalizes the interpretation of classical interval extensions. This framework represent a new formulation of the modal intervals theory: the subsections 2.1, 2.2, 2.3 and 2.4 present a new formulation of some central results of the modal intervals theory while the subsection 2.5 and 2.6 present a new mean-value extension which was obtained thanks to this new framework.

2.1 Description of the problems to be solved

The problem solved by generalized interval extensions generalizes the problem solved by classical interval extensions. As mentioned in introduction, the problem solved by classical interval extensions can be stated in the following way:

Problem 1. Given a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{IR}^n$, we want to compute an interval $\mathbf{z} \in \mathbb{IR}$ that satisfies the quantified proposition

$$\big(\forall x \in \mathbf{x}\big)\big(\exists z \in \mathbf{z}\big)\big(f(x) = z\big).$$

This problem is generalized in the following way:

Problem 2. Given a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathbf{x} \in \mathbb{IR}^n$ and a quantifier Q_{x_k} for each variable x_k we want to compute a quantifier Q_z and an interval $\mathbf{z} \in \mathbb{IR}$ that satisfy the following quantified proposition:

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\mathsf{Q}_z z \in \mathbf{z}\big)\big(\exists x_{\mathcal{E}} \in \mathbf{x}_{\mathcal{E}}\big)\big(f(x) = z\big), \tag{1}$$

where $\mathcal{A} = \{k \mid \mathsf{Q}_{x_k} = \forall\}$ and $\mathcal{E} = \{k \mid \mathsf{Q}_{x_k} = \exists\}$.

In Problem 2, both $\mathcal{A}$ or $\mathcal{E}$ may be empty, the quantified proposition (1) being then easily adapted. Also, the quantifier block $\big(\mathsf{Q}_z z \in \mathbf{z}\big)$ is written at the center of the quantified proposition (1) so that the order AE is kept whatever the quantifier Q_z is.

Example 2. Let us consider $f(x) = x_1 + x_2$, $\mathbf{x} = ([-1, 1], [4, 8])$, $\mathsf{Q}_{x_1} = \forall$ and $\mathsf{Q}_{x_2} = \exists$ so that we want to compute Q_z and $\mathbf{z}$ which satisfy

$$\big(\forall x_1 \in [-1, 1]\big)\big(\mathsf{Q}_z z \in \mathbf{z}\big)\big(\exists x_2 \in [4, 8]\big)\big(z = f(x)\big)$$

One can check that $\mathsf{Q}_z = \forall$ and $\mathbf{z} = [5, 7]$ is a solution to our problem. Trivially, other weaker solutions can be constructed starting from the latter solution, eg. $\mathsf{Q}_z = \forall$ and $\mathbf{z} = [5.5, 6.5]$ because $[5.5, 6.5] \subseteq [5, 7]$, or $\mathsf{Q}_z = \exists$ and $\mathbf{z} = [6.5, 9]$ because $[6.5, 9] \cap [5, 7] \neq \emptyset$, or $\mathsf{Q}_z = \exists$ and $\mathbf{z} = [6, 10]$ because $[6, 10] \cap [5, 7] \neq \emptyset$.

As illustrated by Example 2, two solutions to Problem 2 can be compared in the following way: let $(\mathsf{Q}, \mathbf{x})$ and $(\mathsf{Q}', \mathbf{x}')$ be two such solutions. It is clear that $(\mathsf{Q}, \mathbf{x})$ is more accurate than $(\mathsf{Q}', \mathbf{x}')$ if one of the following conditions is met:

- $\mathsf{Q} = \forall$ and $\mathsf{Q}' = \forall$ and $\mathbf{x} \supseteq \mathbf{x}'$,
- $\mathsf{Q} = \exists$ and $\mathsf{Q}' = \exists$ and $\mathbf{x} \subseteq \mathbf{x}'$,
- $\mathsf{Q} = \forall$ and $\mathsf{Q}' = \exists$ and $\mathbf{x} \cap \mathbf{x}' \neq \emptyset$,

In other cases, the two solutions are not comparable, i.e. they provide complementary informations.

Remark 1. In the case of vector-valued functions, i.e. $f = (f_1, \dots, f_m)$, Problem 2 consists in computing m intervals $\mathbf{z}_k$ and m quantifiers Q_{z_k}. The quantified proposition to be satisfied is then

$$\big(\forall x_{\mathcal{A}} \in \mathbf{x}_{\mathcal{A}}\big)\big(\forall z_{\mathcal{A}'} \in \mathbf{z}_{\mathcal{A}'}\big)\big(\exists z_{\mathcal{E}'} \in \mathbf{z}_{\mathcal{E}'}\big)\big(\exists x_{\mathcal{E}} \in \mathbf{x}_{\mathcal{E}}\big)\big(f(x) = z\big),$$

where $\mathcal{A}' = \{k \mid \mathsf{Q}_{z_k} = \forall\}$ and $\mathcal{E}' = \{k \mid \mathsf{Q}_{z_k} = \exists\}$. A simplified presentation is proposed restricting the formal definitions to real-valued functions.

2.2 Generalized intervals and quantifiers

We now restate Problem 2 using the language of generalized intervals. This will be valuable for many reasons: manipulations of quantified propositions will then be done computing with generalized intervals, leading to efficient computations. Furthermore the structure of generalized intervals is strongly related to Problem 2 and will therefore offer useful properties, eg. the comparison of two solutions of Problem 2 will be related to the inclusion between generalized intervals, some easy rounding process will be available.

A generalized interval $\mathbf{x} \in \mathbb{KR}$ is an interval whose bounds are not constrained to be ordered. For example, $[-1, 1] \in \mathbb{IR}$ is a proper interval and $[1, -1] \in \overline{\mathbb{IR}}$ is an improper one. So, related to a set of reals $\{x \in \mathbb{R} \mid a \leq x \leq b\}$, one can consider two generalized intervals $[a, b]$ and $[b, a]$. It will be convenient to use the operations $\text{dual}\,[a, b] = [b, a]$ and $\text{pro}\,[a, b] = [\min\{a, b\}, \max\{a, b\}]$ to change the proper/improper quality keeping unchanged the underlying set of reals. An inclusion is defined for generalized intervals using the same formal expression than in the context of classical intervals: $\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \leq \underline{\mathbf{x}} \wedge \overline{\mathbf{x}} \leq \overline{\mathbf{y}}$. For example, $[-1, 1] \subseteq [-1.1, 1.1]$, $[1.1, -1.1] \subseteq [1, -1]$ (the inclusion between the underlying sets of real is reversed) and $[2, 0.9] \subseteq [-1, 1]$ (the underlying sets of reals have at least one common point).

Instead of associating both an interval and a quantifier to a variable, we now associate a generalized interval to a variable. On one hand, given a generalized interval $\mathbf{x}_k$ (resp. $\mathbf{z}_k$) associated to x_k (resp. z_k), the domain of x_k (resp. z_k) will be $\text{pro}\,\mathbf{x}_k$ (resp. $\text{pro}\,\mathbf{z}_k$). On the other hand, we choose the following convention to link the proper/improper quality to a quantifier[4]:

- If $\mathbf{x}_k$ is proper ($\underline{\mathbf{x}}_k \leq \overline{\mathbf{x}}_k$) then $\mathsf{Q}_k = \forall$. If $\mathbf{x}_k$ is not proper ($\underline{\mathbf{x}}_k > \overline{\mathbf{x}}_k$) then $\mathsf{Q}_k = \exists$.
- If $\mathbf{z}$ is proper then $\mathsf{Q} = \exists$. If $\mathbf{z}$ is not proper then $\mathsf{Q} = \forall$.

We are now in position to reformulate Problem 2 using generalized intervals:

Problem 3. Given a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{KR}^n$ we want to compute $\mathbf{z} \in \mathbb{KR}$ such that the following quantified proposition is true:

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \mathbf{z}\big)\big(\exists x_{\mathcal{I}} \in \mathbf{x}_{\mathcal{I}}\big)\big(f(x) = z\big), \tag{2}$$

where $\mathcal{P} = \{k \mid \mathbf{x}_k \in \mathbb{IR}\}$, $\mathcal{I} = \{k \mid \mathbf{x}_k \notin \mathbb{IR}\}$ and $\mathsf{Q}_{\mathbf{z}} = \exists$ if $\mathbf{z} \in \mathbb{IR}$ and $\mathsf{Q}_{\mathbf{z}} = \forall$ if $\mathbf{z} \notin \mathbb{IR}$. Such a generalized interval $\mathbf{z}$ is called interpretable w.r.t. f and $\mathbf{x}$—or shortly $(f, \mathbf{x})$-interpretable.

So, when all intervals are proper, we retrieve the interpretation of classical interval extensions $\big(\forall x \in \mathbf{x}\big)\big(\exists z \in \mathbf{z}\big)\big(f(x) = z\big)$. When a proper interval is changed to an improper one, the related quantifier is changed—keeping the quantified proposition in AE-form.

[4] This convention is chosen so as to match the classical interval extensions interpretation when all intervals are proper.

Example 3. From Example 1, $[1, 7]$ and $[0.75, 6.75]$ are $(f, [2, 3])$-interpretable (the classical interpretation is retrieved when intervals are proper). From Example 2, the intervals $[7, 5]$, $[6.5, 5.5]$, $[6.5, 9]$ and $[6, 10]$ are $(+, [-1, 1], [8, 4])$-interpretable.

The $(+, [-1, 1], [8, 4])$-interpretable intervals of Example 3 are related by the following inclusions: $[7, 5] \subseteq [6.5, 5.5] \subseteq [6.5, 9] \subseteq [6, 10]$. We can notice that these four solutions are ordered by decreasing accuracy. In general, the generalized intervals inclusion can be used to compare the accuracy of $(f, \mathbf{x})$-interpretable intervals: if $\mathbf{z}$ and $\mathbf{z}'$ are two $(f, \mathbf{x})$-interpretable intervals related by $\mathbf{z} \subseteq \mathbf{z}'$ then $\mathbf{z}$ is more accurate than $\mathbf{z}'$ (see [2]).

Remark 2. This latter property allows to generalize the rounding process used in the context of classical interval extensions. Indeed, if a $(f, \mathbf{x})$-interpretable interval $\mathbf{z}$ has no floating point representation (see [6]), one can compute it with an outer rounding hence leading to $\mathbf{z}'$ satisfying $\mathbf{z}' \supseteq \mathbf{z}$. So that $\mathbf{z}'$ is also $(f, \mathbf{x})$-interpretable (see Example 5 and [11, 2]).

Finally, on the model of the classical intervals theory, we define generalized interval extensions in the following way:

Definition 1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. A generalized interval function $\mathbf{f} : \mathbb{KR}^n \longrightarrow \mathbb{KR}$ is a generalized interval extension of f if for any $\mathbf{x} \in \mathbb{KR}^n$ the generalized interval $\mathbf{f}(\mathbf{x})$ is $(f, \mathbf{x})$-interpretable.*

2.3 The Kaucher arithmetic

We now define a generalized interval arithmetic which solves Problem 3 in the simple cases of elementary functions: given an elementary function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{KR}^n$, where $n = 1$ or $n = 2$ for elementary functions, we define $f(\mathbf{x})$ as the best[5] generalized interval which is $(f, \mathbf{x})$-interpretable. Thanks to the simplicity of the elementary functions, we can compute their expressions formally (see [11, 2]):

- $\mathbf{x} + \mathbf{y} = [\underline{\mathbf{x}} + \underline{\mathbf{y}}, \overline{\mathbf{x}} + \overline{\mathbf{y}}]$. Also, $\mathbf{x} - \mathbf{y} = [\underline{\mathbf{x}} - \overline{\mathbf{y}}, \underline{\mathbf{x}} - \overline{\mathbf{y}}]$.
- If all involved bounds are positive, $\mathbf{x} \times \mathbf{y} = [\underline{\mathbf{x}} \times \underline{\mathbf{y}}, \overline{\mathbf{x}} \times \overline{\mathbf{y}}]$ (see Table 1 for the other cases which do not match any more the classical expressions). Also, $\mathbf{x}/\mathbf{y} = \mathbf{x} \times (1/\mathbf{y})$ with $1/\mathbf{y} = [1/\overline{\mathbf{y}}, 1/\underline{\mathbf{y}}]$.
- for continuous one variable functions, pro $f(\mathbf{x}) = \{f(x) \mid x \in \text{pro } \mathbf{x}\}$ and both $f(\mathbf{x})$ and $\mathbf{x}$ have the same proper/improper quality, eg. $\sqrt{\mathbf{x}} = [\sqrt{\underline{\mathbf{x}}}, \sqrt{\overline{\mathbf{x}}}]$ for $\underline{\mathbf{x}}, \overline{\mathbf{x}} \geq 0$.

This arithmetic was already proposed in an other context and is today called the *Kaucher arithmetic* (see [8]).

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[5] The smallest in the sense of the generalized intervals inclusion, i.e. the largest domain when $\mathsf{Q} = \forall$ and the smallest domain when $\mathsf{Q} = \exists$—see [2] for more details.

Table 1. The Kaucher multiplication

$\mathbf{x} \times \mathbf{y}$	$\mathbf{y} \in \mathcal{P}$	$\mathbf{y} \in \mathcal{Z}$	$\mathbf{y} \in -\mathcal{P}$	$\mathbf{y} \in \text{dual } \mathcal{Z}$
$\mathbf{x} \in \mathcal{P}$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$
$\mathbf{x} \in \mathcal{Z}$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\}, \max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}]$	$[\overline{\mathbf{x}}\,\underline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	0
$\mathbf{x} \in -\mathcal{P}$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\underline{\mathbf{y}}]$	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$
$\mathbf{x} \in \text{dual } \mathcal{Z}$	$[\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}]$	0	$[\overline{\mathbf{x}}\,\overline{\mathbf{y}}, \underline{\mathbf{x}}\,\overline{\mathbf{y}}]$	$[\max\{\underline{\mathbf{x}}\,\underline{\mathbf{y}}, \overline{\mathbf{x}}\,\overline{\mathbf{y}}\}, \min\{\underline{\mathbf{x}}\,\overline{\mathbf{y}}, \overline{\mathbf{x}}\,\underline{\mathbf{y}}\}]$

where $\mathcal{P} = \{\mathbf{x} \in \mathbb{KR} | 0 \le \underline{\mathbf{x}} \wedge 0 \le \overline{\mathbf{x}}\}$, $-\mathcal{P} = \{\mathbf{x} \in \mathbb{KR} | 0 \ge \underline{\mathbf{x}} \wedge 0 \ge \overline{\mathbf{x}}\}$, $\mathcal{Z} = \{\mathbf{x} \in \mathbb{KR} | \underline{\mathbf{x}} \le 0 \le \overline{\mathbf{x}}\}$ and dual $\mathcal{Z} = \{\mathbf{x} \in \mathbb{KR} | \underline{\mathbf{x}} \ge 0 \ge \overline{\mathbf{x}}\}$.

Example 4. The Kaucher addition $[0, 2] + [7, 8] = [7, 10]$ coincides with the classical intervals addition and is interpreted in the same way:

$$\big(\forall x_1 \in [0, 2]\big)\big(\forall x_2 \in [7, 8]\big)\big(\exists z \in [7, 10]\big)\big(z = x_1 + x_2\big).$$

The Kaucher addition $[0, 2] + [8, 7] = [8, 9]$ is interpreted as

$$\big(\forall x_1 \in [0, 2]\big)\big(\exists z \in [8, 9]\big)\big(\exists x_2 \in [7, 8]\big)\big(z = x_1 + x_2\big),$$

while the Kaucher addition $[0, 2] + [8, 4] = [8, 6]$ is interpreted as

$$\big(\forall x_1 \in [0, 2]\big)\big(\forall z \in [6, 8]\big)\big(\exists x_2 \in [4, 8]\big)\big(z = x_1 + x_2\big).$$

Example 5. The Kaucher square $[2, 3]^2 = [4, 9]$ is interpreted by

$$\big(\forall x \in [2, 3]\big)\big(\exists z \in [4, 9]\big)\big(z = x^2\big).$$

Notice that an outer rounding of this operation would lead for example to $[3.9, 9.1] \supseteq [2, 3]^2$ which is still interpretable (the involved domain has been enlarged by outer rounding). Now $[3, 2]^2 = [9, 4]$ is interpreted by

$$\big(\forall z \in [4, 9]\big)\big(\exists x \in [2, 3]\big)\big(z = x^2\big).$$

In this case, an outer rounding of this operation would lead for example to $[8.9, 4.1] \supseteq [3, 2]^2$ which is still interpretable (the involved domain has been retracted by outer rounding because the rounded interval is improper).

2.4 A simplified interpretation of the generalized interval evaluation

Let $\mathsf{f}(x_1, \ldots, x_n)$ be an expression for a function f where each variable appears only once. Then $\mathsf{f}(\mathbf{x})$, computed for $\mathbf{x} \in \mathbb{KR}^n$ using the Kaucher arithmetic, raises an interval which is $(f, \mathbf{x})$-interpretable. The following example illustrates this property:

Example 6. Let f be the function whose expression is $\mathsf{f}(x,y,u) = u(x+y)$ and $\mathbf{x},\mathbf{u} \in \mathbb{IR}$ and $\mathbf{y} \in \overline{\mathbb{IR}}$ (so that $\mathsf{Q}_x = \mathsf{Q}_u = \forall$ and $\mathsf{Q}_y = \exists$). Then we compute $\mathbf{z} := \mathsf{f}(\mathbf{x},\mathbf{y},\mathbf{u})$, i.e. $\mathbf{x}+\mathbf{y}=\mathbf{t}$ and $\mathbf{u}\times\mathbf{t}=\mathbf{z}$. On one hand, if $\mathbf{t} \in \mathbb{IR}$ then

$$\begin{array}{l} \mathbf{t}=\mathbf{x}+\mathbf{y} \text{ is interpreted by } \big(\forall x \in \mathbf{x}\big)\big(\exists t \in \mathbf{t}\big)\big(\exists y \in \text{pro } \mathbf{y}\big)\big(t = x+y\big), \\ \mathbf{z}=\mathbf{u}\times\mathbf{t} \text{ is interpreted by } \big(\forall u \in \mathbf{u}\big)\big(\forall t \in \mathbf{t}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \mathbf{z}\big)\big(z = u\,t\big). \end{array}$$

On the other hand, if $\mathbf{t} \in \overline{\mathbb{IR}}$ then

$$\begin{array}{l} \mathbf{t}=\mathbf{x}+\mathbf{y} \text{ is interpreted by } \big(\forall x \in \mathbf{x}\big)\big(\forall t \in \text{pro } \mathbf{t}\big)\big(\exists y \in \text{pro } \mathbf{y}\big)\big(t = x+y\big), \\ \mathbf{z}=\mathbf{u}\times\mathbf{t} \text{ is interpreted by } \big(\forall u \in \mathbf{u}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \mathbf{z}\big)\big(\exists t \in \text{pro } \mathbf{t}\big)\big(z = u\,t\big). \end{array}$$

Finally, in both cases, the following quantified proposition is entailed:

$$\big(\forall x \in \mathbf{x}\big)\big(\forall u \in \mathbf{u}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \mathbf{z}\big)\big(\exists y \in \text{pro } \mathbf{y}\big)\big(\exists t \in \text{pro } \mathbf{t}\big)\big(z = u\,t \wedge t = x+y\big)$$

Therefore, noticing that $\big(\exists t \in \text{pro } \mathbf{t}\big)\big(z = u\,t \wedge t = x+y\big) \Longrightarrow z = f(x,y,u)$, we come to the conclusion that $\mathbf{z}$ is $(f,\mathbf{x},\mathbf{y},\mathbf{u})$-interpretable.

The argumentation presented in Example 6 is easily generalized to any expression containing one occurrence of each variable and to any interval arguments. If the expression contains several occurrences of some variables then constructing a $(f,\mathbf{x})$-interpretable interval needs further developments. A generalized interval natural extension has been defined in [2] in the following way: a new expression g is built from the expression f by inserting an operation pro before all but one occurrences of each variable. Then, $\mathsf{g}(\mathbf{x})$ is $(f,\mathbf{x})$-interpretable. The generalized interval extension $\mathsf{g}(\mathbf{x})$ is called a generalized interval natural extension of f because if all interval arguments are proper then the operations pro has no influence and the classical interval natural extension is retrieved.

Example 7. Let us consider the function $f(x,y) = xy + x(x+y)$. There are several ways to insert an operation pro before all but one occurrences of each variable:

$$\begin{array}{rcl} \mathbf{x}\mathbf{y} + (\text{pro } \mathbf{x})(\text{pro } \mathbf{x} + \text{pro } \mathbf{y}) & ; & \mathbf{x}(\text{pro } \mathbf{y}) + (\text{pro } \mathbf{x})(\text{pro } \mathbf{x} + \mathbf{y}) \\ (\text{pro } \mathbf{x})\mathbf{y} + \mathbf{x}(\text{pro } \mathbf{x} + \text{pro } \mathbf{y}) & ; & (\text{pro } \mathbf{x})(\text{pro } \mathbf{y}) + \mathbf{x}(\text{pro } \mathbf{x} + \mathbf{y}) \\ (\text{pro } \mathbf{x})\mathbf{y} + (\text{pro } \mathbf{x})(\mathbf{x} + \text{pro } \mathbf{y}) & ; & (\text{pro } \mathbf{x})(\text{pro } \mathbf{y}) + (\text{pro } \mathbf{x})(\mathbf{x} + \mathbf{y}) \end{array}$$

These interval functions are the generalized interval natural extensions of f.

2.5 An original generalized interval mean-value extension

Let us consider $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ differentiable, $\mathbf{x} \in \mathbb{KR}^n$ and $\Delta_i \in \mathbb{IR}$ such that

$$\left\{ \frac{\partial f}{\partial x_i}(x) \mid x \in \text{pro } \mathbf{x} \right\} \subseteq \Delta_i .$$

The Δ_i can be computed using an interval evaluation of the expressions of f derivatives. Then, the following interval is $(f,\mathbf{x})$-interpretable:

$$f^{MV}(\mathbf{x}) := f(\tilde{x}) + \sum_{k=1}^{n} \Delta_k \times (\mathbf{x}_k - \tilde{x}_k) \quad \text{where} \quad \tilde{x} \in \text{pro } \mathbf{x}.$$

Let us justify this statement: on one hand, $\mathbf{z} := f^{MV}(\mathbf{x})$ is the generalized interval evaluation of the function $g(x,\delta) = f(\tilde{x}) + \sum \delta_k(x_k - \tilde{x}_k)$ for $\Delta \in \mathbb{IR}^n$ and $\mathbf{x} \in \mathbb{KR}^n$. As the expression of g contains only one occurrence of each variable x_k or δ_k, the following quantified proposition is true:

$$\big(\forall \delta \in \Delta\big)\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \text{pro } \mathbf{z}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \mathbf{x}_{\mathcal{I}}\big)\big(z = g(x,\delta)\big),$$

where $\mathcal{P} = \{k \mid \mathbf{x}_k \in \mathbb{IR}\}$ and $\mathcal{I} = \{k \mid \mathbf{x}_k \notin \mathbb{IR}\}$. On the other hand, the mean-value theorem (see eg. [16]) entails the quantified proposition

$$\big(\forall x \in \mathbf{x}\big)\big(\exists \delta \in \Delta\big)\big(f(x) = g(x,\delta)\big).$$

It can be proved (see [1]) that the conjunction of these latter two quantified propositions entails the quantified proposition

$$\big(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}}\big)\big(\mathsf{Q}_{\mathbf{z}} z \in \text{pro } \mathbf{z}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \mathbf{x}_{\mathcal{I}}\big)\big(\exists \delta \in \Delta\big)\big(z = g(x,\delta) \wedge f(x) = g(x,\delta)\big).$$

Finally, as $\big(\exists \delta \in \Delta\big)\big(z = g(x,\delta) \wedge f(x) = g(x,\delta)\big)$ implies $z = f(x)$, the interval $\mathbf{z}$ is $(f,\mathbf{x})$-interpretable. The generalized interval extension $f^{MV}(\mathbf{x})$ is called a generalized interval mean-value extension because of the similitude with the classical mean-value extension. It has a quadratic order of convergence and is therefore expected to raise much better results than the natural AE-extension in realistic situations (see [1]).

Example 8. Let us consider $f(x) = x_1(x_1 - x_2)$, $\mathbf{x}_1 = [4,2]$ and $\mathbf{x}_2 = [0,1]$ (so that $\mathsf{Q}_{x_1} = \exists$ and $\mathsf{Q}_{x_2} = \forall$). We can use $\tilde{x} = \text{mid } \mathbf{x}$, $\Delta_1 = 2(\text{pro } \mathbf{x}_1) - \mathbf{x}_2 = [3,8]$ and $\Delta_2 = -(\text{pro } \mathbf{x}_1) = [-4,-2]$. The generalized interval mean-value extension raises $f^{MV}(\mathbf{x}) = [8.5, 6.5]$. The following quantified proposition is therefore validated:

$$\big(\forall x_2 \in [0,1]\big)\big(\forall z \in [6.5, 8.5]\big)\big(\exists x_1 \in [2,4]\big)\big(z = f(x)\big).$$

2.6 Generalized interval mean-value extension for vector-valued functions

The situation is more complicated when dealing with vector-valued functions. Let $F(a,x) = (f_1, f_2) : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\mathbf{a} \in \mathbb{IR}$ and $\mathbf{x} \in \overline{\mathbb{IR}}^2$ (so that $\mathsf{Q}_a = \forall$ and $\mathsf{Q}_{x_1} = \mathsf{Q}_{x_2} = \exists$). We want to compute a generalized interval $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$ which satisfy

$$\big(\forall a \in \mathbf{a}\big)\begin{matrix}\big(\mathsf{Q}_{\mathbf{z}_1}\, z_1 \in \text{pro } \mathbf{z}_1\big)\\ \big(\mathsf{Q}_{\mathbf{z}_2}\, z_2 \in \text{pro } \mathbf{z}_2\big)\end{matrix}\big(\exists x_1 \in \text{pro } \mathbf{x}_1\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z = F(a,x)\big)^{6} \quad (3)$$

using the generalized interval mean-value extensions of the functions f_1 and f_2. A way to obtain the wanted interpretation is to compute $\mathbf{z}_1 = f_1^{MV}(\mathbf{a}, \mathbf{x}_1, \text{pro } \mathbf{x}_2)$ and $\mathbf{z}_2 = f_2^{MV}(\mathbf{a}, \text{pro } \mathbf{x}_1, \mathbf{x}_2)$. Hence we get the following interpretations:

$$\begin{matrix}\big(\forall a \in \mathbf{a}\big)\big(\forall x_2 \in \text{pro } \mathbf{x}_2\big)\big(\mathsf{Q}_{\mathbf{z}_1}\, z_1 \in \text{pro } \mathbf{z}_1\big)\big(\exists x_1 \in \text{pro } \mathbf{x}_1\big)\big(z_1 = f_1(a,x)\big)\\ \big(\forall a \in \mathbf{a}\big)\big(\forall x_1 \in \text{pro } \mathbf{x}_1\big)\big(\mathsf{Q}_{\mathbf{z}_2}\, z_2 \in \text{pro } \mathbf{z}_2\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z_2 = f_2(a,x)\big)\end{matrix}$$

[6] The blocks $\big(\mathsf{Q}_{\mathbf{z}_1}\, z_1 \in \text{pro } \mathbf{z}_1\big)$ and $\big(\mathsf{Q}_{\mathbf{z}_2}\, z_2 \in \text{pro } \mathbf{z}_2\big)$ must be placed in order to obtain a quantified proposition in the AE-form.

It can be proved that the latter two quantified propositions entail the quantified proposition (3), i.e. that $\mathbf{z}$ is $(F, \mathbf{a}, \mathbf{x})$-interpretable (see [1]). We denote the generalized interval function

$$\big(\, f_1^{MV}(\mathbf{a}, \mathbf{x}_1, \text{pro } \mathbf{x}_2)\,,\; f_2^{MV}(\mathbf{a}, \text{pro } \mathbf{x}_1, \mathbf{x}_2)\,\big)^T$$

by $F^{MV}(\mathbf{a}, \mathbf{x})$.

Remark 3. The other choice

$$\mathbf{z}_1 = f_1^{MV}(\mathbf{a}, \text{pro } \mathbf{x}_1, \mathbf{x}_2) \;\text{ and }\; \mathbf{z}_2 = f_2^{MV}(\mathbf{a}, \mathbf{x}_1, \text{pro } \mathbf{x}_2),$$

which leads to the interpretations

$$\begin{array}{c}
\big(\forall a \in \mathbf{a}\big)\big(\forall x_1 \in \text{pro } \mathbf{x}_1\big)\big(\mathsf{Q}_{\mathbf{z}_1}\, z_1 \in \text{pro } \mathbf{z}_1\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z_1 = f_1(a, x)\big) \\
\big(\forall a \in \mathbf{a}\big)\big(\forall x_2 \in \text{pro } \mathbf{x}_2\big)\big(\mathsf{Q}_{\mathbf{z}_2}\, z_2 \in \text{pro } \mathbf{z}_2\big)\big(\exists x_1 \in \text{pro } \mathbf{x}_1\big)\big(z_2 = f_2(a, x)\big),
\end{array}$$

would also be correct. However, the generalized interval mean-value extension for vector-valued functions will usually be coupled to a preconditioning process which changes the function F into a near identity function $G = CF$, where C is the preconditioning real matrix. In this situation, the first choice will be more efficient. Finally, the naive computations

$$\mathbf{z}_1 = f_1^{MV}(\mathbf{a}, \mathbf{x}_1, \mathbf{x}_2) \;\text{ and }\; \mathbf{z}_2 = f_2^{MV}(\mathbf{a}, \mathbf{x}_1, \mathbf{x}_2)$$

leads to the following interpretations:

$$\begin{array}{c}
\big(\forall a \in \mathbf{a}\big)\big(\mathsf{Q}_{\mathbf{z}_1}\, z_1 \in \text{pro } \mathbf{z}_1\big)\big(\exists x_1 \in \text{pro } \mathbf{x}_1\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z_1 = f_1(a, x)\big) \\
\big(\forall a \in \mathbf{a}\big)\big(\mathsf{Q}_{\mathbf{z}_2}\, z_2 \in \text{pro } \mathbf{z}_2\big)\big(\exists x_1 \in \text{pro } \mathbf{x}_1\big)\big(\exists x_2 \in \text{pro } \mathbf{x}_2\big)\big(z_2 = f_2(a, x)\big).
\end{array}$$

However, the conjunction of the latter two quantified propositions does not imply (3) in general.

3 Application of the generalized interval mean-value extension

A two arms parallel robot can be described by the vector-valued function

$$r = F(a, x) \;\text{ with }\; r_1 = \sqrt{x_1^2 + x_2^2} \;\text{ and }\; r_2 = \sqrt{(x_1 - a)^2 + x_2^2},$$

where r_k, x_k and a are respectively the commands, the coordinates of the working point and a parameter (see [15]). Given $\mathbf{x} = ([3.9, 4.1], [3.9, 4.1])^T$ and $\mathbf{a} = [9.99, 10.01]$, we want to compute a set of command values each of them leading the working point inside $\mathbf{x}$, this whatever is $a \in \mathbf{a}$. Formally, we want to compute some $S \subseteq \mathbb{R}^2$ which satisfies

$$\big(\forall r \in S\big)\big(\forall a \in \mathbf{a}\big)\big(\exists x \in \mathbf{x}\big)\big(r = F(a, x)\big).$$

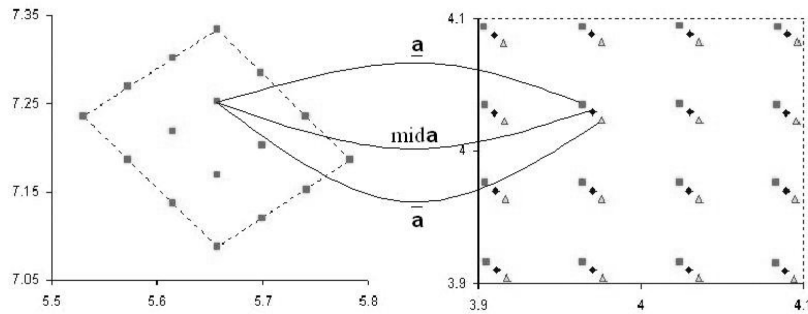

Fig. 1. Computation of the position of the working point for 64 samples.

A preconditioning is needed[7] so we consider the equation $Cr = CF(a,x) \iff r = F(a,x)$ where C is the usual midpoint inverse preconditioning matrix. The Jacobian matrix of F is

$$J(a,x) := \begin{pmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

and the one of CF is $CJ(a,x)$. In this situation, the interval evaluation of the Jacobian matrix of CF is

$$\mathbf{A} \approx C \begin{pmatrix} [0,0] & [.67,.74] & [.67,.74] \\ [.80,.86] & [-.86,-.80] & [.53,.58] \end{pmatrix} \approx \begin{pmatrix} [-.62,-.58] & [.96,1.04] & [-.04,.04] \\ [.58,.62] & [-.05,.05] & [.95,1.05] \end{pmatrix}$$

and it satisfies $\big(\forall a \in \mathbf{a}\big)\big(\forall x \in \mathbf{x}\big)\big(CJ(a,x) \in \mathbf{A}\big)$. We now evaluate the generalized interval mean-value extension of CF for the intervals $\mathbf{a} \in \mathbb{IR}$ and $(\text{dual } \mathbf{x}) \in \overline{\mathbb{IR}}^2$ so that $\mathsf{Q}_a = \forall$ and $\mathsf{Q}_{x_1} = \mathsf{Q}_{x_2} = \exists$ (a simple implementation of the Kaucher arithmetic in C was used):

$$\mathbf{u} := (CF)^{MV}(\mathbf{a}, \text{dual } \mathbf{x}) \approx ([-1.91, -2.09], [10.08, 9.90]).$$

As $\mathbf{u}$ is improper, this computation is interpreted by

$$\big(\forall u \in \text{pro } \mathbf{u}\big)\big(\forall a \in \mathbf{a}\big)\big(\exists x \in \mathbf{x}\big)\big(u = Cf(a,x)\big).$$

Finally, for any command r in the parallelepiped $\{C^{-1}u \mid u \in \text{pro } \mathbf{u}\}$—left hand side graphic of Figure 1—and any $a \in \mathbf{a}$, the working point x lies inside $\mathbf{x}$. This is illustrated by Figure 1 where the system $r = F(a,x)$ has been solved for 64 points (a,r) satisfying $a \in \mathbf{a}$ and $r \in \{C^{-1}u \mid u \in \text{pro } \mathbf{u}\}$. We can see that the parallelepiped is almost optimal: some solutions are very close to each side of the box $\mathbf{x}$.

[7] See [3, 2] for more details about the preconditioning which is used.

4 Related work

If the equation $r = F(a, x)$ can be written under the form $x = H(a, r)$, then the classical interval analysis provides a test for a box $\mathbf{r}$ to satisfy the problem: indeed, if $\mathbf{H}(\mathbf{a}, \mathbf{r}) \subseteq \mathbf{x}$, where $\mathbf{H}$ is a classical interval extension of H, then the quantified proposition $\big(\forall r \in \mathbf{r}\big)\big(\forall a \in \mathbf{a}\big)\big(\exists x \in \mathbf{x}\big)\big(x = H(a, r)\big)$ is true and thus the box $\mathbf{r}$ is proved to be a solution of the problem of Section 3. Then, a bisection algorithm can compute an accurate inner approximation of the wanted set.

The quantifier elimination (see [14, 7]) can be used to changed the problem of Section 3 into a quantifier free problem which could be solved by some basic interval bisection algorithm. However, the complexity of the quantifier elimination is known to restrict its application to small problems. The following code was executed with Mathematica5.1 on a PentiumVI 2Ghz with a 512Mo memory:

```
Resolve[ r1 >= 0 && r2 >= 0 &&
  ForAll[a , a\[Element] Reals && 10 - 1/100 <= a <= 10 + 1/100 ,
    Exists[x1 , x1\[Element] Reals && 4 - 1/10 <= x1 <= 4 + 1/10 ,
      Exists[x2 , x2\[Element] Reals && 4 - 1/10 <= x2 <= 4 + 1/10 ,
        r1^2 == x1^2 + x2^2 && r2^2==(x1 - a)^2 + x2^2
      ]
    ]
  ], Reals]
```

Mathematica didn't provide any solution after 20 minutes of computations.

No other numerical algorithm has been yet proposed to solve the problem of Section 3. The techniques dedicated to quantified inequality constraints (see [10, 22]) cannot solve this problem. Other techniques (see [21, 19]) can handle quantified constraints with one equality, i.e. $f(x) = 0$ with $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ a real-valued function. These techniques can be easily extended to several equality constraints if those constraints do not share any existentially quantified variables. But the problem of Section 3 contains some existentially quantified variables which are shared between the two equations.

5 Conclusion

The modal intervals theory allows to change the problem of validation of some quantified propositions into the computation of generalized interval functions. Thanks to our simpler formulation of the modal intervals theory, we have defined a generalized interval mean-value extension whose evaluation validates some quantified propositions. An example problem which is not well solved today has been proposed and one evaluation of the generalized interval mean-value extension has provided a sharp solution. So as to be efficient, the mean-value extensions have to be applied to small enough intervals so that the involved interval Jacobian matrix is regular. Future works will consist in elaborating a bisection algorithm so as to deal with larger intervals.

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