Verified Projection of the Solution Set of Parametric Real Systems

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Abstract. Projection of the solution set of a conjunction of real constraints has numerous applications : For example verifying existence of a solution for a parametric $n \times n$ system for every values of the parameters; or extracting informations of a system from low dimensional projections of its solution set. We study parametric systems of the following form : $F(p, x) = \theta$ with $F : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ being differentiable. We study the projection into the subspace $\mathbb{R}^p$ of its solution set.
Usual methods are able to prune the search space $\mathbb{R}^p$ from points which are out of the projection, hence leading to an outer estimate of the wanted projection. We propose the theoretical basis for a numerical method which will give a reliable inner estimate of the wanted projection.

1 Notations

A n-dimensional interval – also named box – is denoted in boldface $\mathbf{x} = \mathbf{x}_1 \times \cdots \times \mathbf{x}_n \in \mathbb{IR}^n$.
Interval evaluation of a real-valued function is noted $f(\mathbf{x})$. It can be any interval extensions of the function f – optimal extension, natural extension, centered form... Its characteristic properties are : 1 $range_{\mathbf{x}}(f) \subset f(\mathbf{x})$, monotonicity, $\mathbf{x} \subset \mathbf{y} \Rightarrow f(\mathbf{x}) \subset f(\mathbf{y})$ 3 continuity, see **??** for details – properties 2 and 3 being asked for section 5.
A real vector is denoted by $x \in \mathbb{R}^n$ and its components by $x_k \in \mathbb{R}$. The null vector is denoted θ whatever is the dimension of the space.
A real-vector-valued function is denoted by $F : \mathbb{R}^n \to \mathbb{R}^m$ and its components are noted $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in [1..m]$, i.e. $F = (f_1, \ldots, f_n)$.
A parametric system is denoted by a vector-valued function : $F : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$. It can also be denoted by a parametric form, for any $p \in P \subset \mathbb{R}^p$, $F(p,.) : \mathbb{R}^n \to \mathbb{R}^n$; $F(p,.)(x) = F(p,x)$.
The Jacobi matrix of a real-vector-valued function $F : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ is denoted by $\frac{\partial F}{\partial(p,x)}(p,x) \in \mathbb{R}^{n\times(p+n)}$ or by $F'(p,x)$ if no confusion is possible. As $F(p,.)$ is considered as $n \times n$ system, its Jacobi matrix is square and is denoted by: $\frac{\partial F}{\partial x}(p,x) \in \mathbb{R}^{n\times n}$.

Example 1. Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$; $F(p,x) = \begin{pmatrix} p_1 + x_1 + p_2 x_2 \\ p_1 + x_1 - p_2 x_2 \end{pmatrix}$.
Following matrix is F Jacobi matrix – variables are ordered as p_1, p_2, x_1 and x_2:
$F'(p,x) = \begin{pmatrix} 1\ ,\ x_2\ ,\ 1\ ,\ p_2 \\ 1\ ,\ -x_2\ ,\ 1\ ,\ -p_2 \end{pmatrix}$.
The Jacobi matrix of the parametric form of the function is:
$\frac{\partial F}{\partial x}(p,x) = \begin{pmatrix} 1\ ,\ p_2 \\ 1\ ,\ -p_2 \end{pmatrix}$.

A matrix is denoted by $M = (m_{ij}) \in \mathbb{R}^{n\times m}$. An interval matrix is denoted using boldface: $\mathbf{M} \in \mathbb{IR}^{n\times m}$.

2 introduction

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function, and F' its Jacobi matrix. The equation $F(x) = \theta$ stands for a $n \times n$ system of equations, i.e. $f_k(x) = 0$ for all $k \in [1..n]$. In the case where the solution x of $F(x) = \theta$ is regular – i.e. $\det(F'(x)) \neq 0$ – a lot of ways can be used to show its existence inside a given little box around it. They usually rely on Brouwer's theorem, Miranda's theorem: See [13] and [12] for clear and short descriptions of main methods. For singular solutions, improved methods have to be used, for example [5] propose to use the topological degree of F.
We are going to generalize the use of The Miranda theorem to the parametric systems: In the case of systems which come from real world, parameters are usually not exactly known. Intervals are wildly used to bound those uncertainties.

In this situation, one wants to show that the system has a solution for every parameters values inside the parameters box. To this end, we propose the following definition, which is restricted to square parametric systems:

Definition 1 (Verified Projection Problem).
Let $F : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ *be differentiable inside a given box* $\mathbf{p} \times \mathbf{x} \subset \mathbb{R}^p \times \mathbb{R}^n$*. Define the verified projection* $proj(F = \theta, \mathbf{p}, \mathbf{x})$ *as the following set of parameters values:* $proj(F = \theta, \mathbf{p}, \mathbf{x}) = \left\{p \in \mathbf{p} | \exists x \in \mathbf{x}, F(p, x) = 0\right\}$*. The verified projection problem consists in extracting a subset* S *of* $proj([= \theta, F)][\mathbf{p}, \mathbf{x}]$*, ie* S *is such that*

$$\forall p \in S, \exists x \in \mathbf{x}, F(p, x) = \theta$$

Remark 1. Differentiability is asked for so as to be able to precondition the system. See section 4.

Remark 2. Due to the generality of the verified projection problem, its resolution can be used in many other contexts. See example 8.

Next example is an instantiation of the verified projection problem.

Example 2. Let $F : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2$; $F(p, x) = \begin{pmatrix} p^2 + x_1^2 + x_2^2 - 1 \\ p + x_1 + x_2 \end{pmatrix}$ and $\mathbf{p} \times \mathbf{x} = [-2, 2]^3$. $F = \theta$ is the intersection of a 2-dimensional sphere and a 2-dimensional plan. Following subsets of $\mathbf{p}$ are solutions of :

- $S = \emptyset$ is a trivial solution,
- $S = [-0.1, 0.1] \subset \mathbf{p}$ is a non-empty solution because $\forall p \in S, \exists x \in \mathbf{x}, F(p, x) = \theta$,
- $S = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ can be shown to be the optimal solution by geometric considerations – $F(p, x) = \theta \Leftrightarrow (x_1^2 + x_2^2 = 1 - p^2 \wedge x_2 = -x_1 - p)$, which is a simple intersection problem between a circle and a line.

In this paper, we propose to extend one of the method which works for non parametric systems to the parametric case. A sufficient condition for a parameters box to be included inside $proj(F = \theta, \mathbf{p}, \mathbf{x})$ is designed from a weak version of the parametric Miranda theorem. We show that preconditioning is needed in the general case, and how a parametric system has to be preconditioned.

State of the Art In the case of polynomial systems with integer - or rational - coefficients, cylindrical algebraic decomposition is a formal method which can be used to build a non quantified system of constraints inside the free variable space - ie parameters space in our case. This quantifier free constraint system has a solution set which is the same than the one of the studied verified projection problem. Its solutions can be approach with usual techniques. Complexity of its transformation is doubly exponential and only small problems can be treated. See [7] for an introduction to cylindrical algebraic decomposition.
In [10], the author proposed the basis for a general numerical method to solve quantified constraints with equality predicates. This method relies on the parametric Miranda theorem proposed in [2]. As a verified projection problem is a

simple quantified constraints satisfaction problem - universal quantifiers precede existential ones - the proposed method can solve them. In spite of this potential effectiveness, on one hand, to our knowledge, no practical applications have been designed using the proposed method. On the other hand, inside the proof of Theorem 1 - pages 3 to 5, the parametric Miranda theorem is used to build a continuous real-valued function, whereas the original formulation of the theorem proves the existence of a c-continuous set-valued function. Therefore, theoretical background of this method is hard to understand.
In a recent paper [11], the same author proposed a method for the 1-dimensional case, relying on the Intermediate Value Theorem. The following elementary step can be extracted from the "check" part of this method : Let a function $f : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous inside $\mathbf{p} \times \mathbf{x}$ - $B_x \times B_y$ in [11]. If one can rise two boxes $\mathbf{d}_1 \subset X$ and $\mathbf{d}_2 \subset X$ such that $range_{P \times \mathbf{d}_1}(f) \leq 0$ and $range_{\mathbf{p} \times \mathbf{d}_2}(f) \geq 0$, then he can use the Intermediate Value Theorem at each point $p \in P$ in order to conclude that $\forall p \in P, \exists x \in X, f(p,x) = 0$ - this is a particular case of our method which is illustrated by example 4. This check can be achieved using interval extensions of f or, as proposed in [11], by pruning $(f > 0, \mathbf{p} \times \mathbf{d}_1)$ and $(f < 0, \mathbf{p} \times \mathbf{d}_2)$ with sound contractors.

Organization of the paper In section 3, weak parametric Miranda theorem is presented and its application to verified projection problems is shown on examples. In section 4, a general sufficient condition is presented for verified projection problems based on weak parametric Miranda theorem applied to the preconditioned problem. The use of conservative contractors is presented as an answer to some stability problems of the application of the proposed sufficient condition within branch and bound algorithms. Section 5 gives a sufficient for a positive application of previous method. Finally, section 6 gives conclusions and describes the future work to be done.

3 Weak parametric Miranda theorem and its basic application to VPP

3.1 Miranda theorem

We shortly recall how Miranda's theorem – corollary 5.3.8 page 197 in [2] – can show the existence of a solution for the system $F(x) = \theta$ with $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Theorem 1 (Miranda theorem). *Let $F : D_F \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous in $\mathbf{x} \subset D_F$, $\mathbf{x} = [a_1, b_1] \times \ldots \times [a_n, b_n]$, $F = (f_1, \ldots, f_n)^T$. If for all i, $1 \leq i \leq n$,*
($x \in \mathbf{x}$ and $x_i = a_i$) $\Rightarrow f_i(x) \leq 0$, and
($x \in \mathbf{x}$ and $x_i = b_i$) $\Rightarrow f_i(x) \geq 0$,
– or reversed inequalities and up to a permutation of F components – then,

$$\exists x^* \in \mathbf{x}, F(x^*) = \theta$$

We now illustrate the use of Miranda's theorem to prove the existence of a solution of a $n \times n$ non-linear system – with $n = 2$.

Example 3. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$; $F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -\sin(x_1) + 4x_2 \\ -x_1 + \sin(2x_2) \end{pmatrix}$ and $\mathbf{x} = \mathbf{x}_1 \times \mathbf{x}_2 = [-10, 10]^2$. Figure 1 shows the implicit curves of $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$. On can easily check that $F(0, 0) = \theta$. So as to prove this with numerical tools, Miranda theorem is applied in the following way :

- Following two interval evaluations : $f_1(\mathbf{x}_1, [-10, -10]) = [-41, -39] < 0$ and $f_1(\mathbf{x}_1, [10, 10]) = [39, 41] > 0$. This let one conclude that the implicit graph of $f_1(x_1, x_2) = 0$ goes continuously from $x_1 = -10$ to $x_1 = 10$ – the horizontal sinusoid on figure 1.
- Following two interval evaluations : $f_2([-10, -10], \mathbf{x}_2) = [9, 11] > 0$ and $f_2([10, 10], \mathbf{x}_2) = [-11, -9] < 0$. This let one conclude that the implicit graph of $f_2(x_1, x_2) = 0$ goes continuously from $x_2 = -10$ to $x_2 = 10$ – the vertical sinusoid on figure 1.
- Eventually, from the four latter interval evaluations, Miranda theorem let one conclude that $\exists x^* \in [-10, 10] \times [-10, 10], F(x^*) = \theta$ – the horizontal implicit curve has to cross the vertical one, this intersection being a solution of the system.

Remark 3. This example is chosen so that Miranda theorem cannot be applied to a little square box closer to the solution, for example $[-10^{-2}, 10^{-2}]^2$ – see example 7 and figure 3 for details.

3.2 Weak parametric Miranda theorem

We now propose the following weak parametric Miranda theorem with few examples illustrating its application to verified projection problems.

Remark 4. The word "weak" comes from the comparison of this theorem with the parametric Miranda theorem proposed by Neumaier - theorem 5.3.7 page 195 of [2]. This latter is much more powerful than the one we propose here as shown by the following two implications which are in fact trivial : parametric Miranda theorem ⇒ Miranda theorem ⇒ weak parametric Miranda theorem.

Theorem 2 (weak parametric Miranda theorem). *Let $F : D_F \subset \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in $\mathbf{p} \times \mathbf{x} \subset D_F$, $\mathbf{x} = [a_1, b_1] \times \ldots \times [a_n, b_n]$ and $F = (f_1, \ldots, f_n)^T$. If for all i, $1 \le i \le n$,*
($p \in \mathbf{p}$ and $x \in \mathbf{x}$ and $x_i = a_i$) $\Rightarrow f_i(p, x) \le 0$, and
($p \in \mathbf{p}$ and $x \in \mathbf{x}$ and $x_i = b_i$) $\Rightarrow f_i(p, x) \ge 0$,
– or reversed inequalities and up to a permutation of F components – then,

$$\forall p \in \mathbf{p}, \exists x \in \mathbf{x}, f(p, x) = \theta$$

Proof. The proof is trivial as the hypothesis let one apply Miranda theorem to $F(p, .)(x) = F(p, x)$ for each $p \in \mathbf{p}$. □

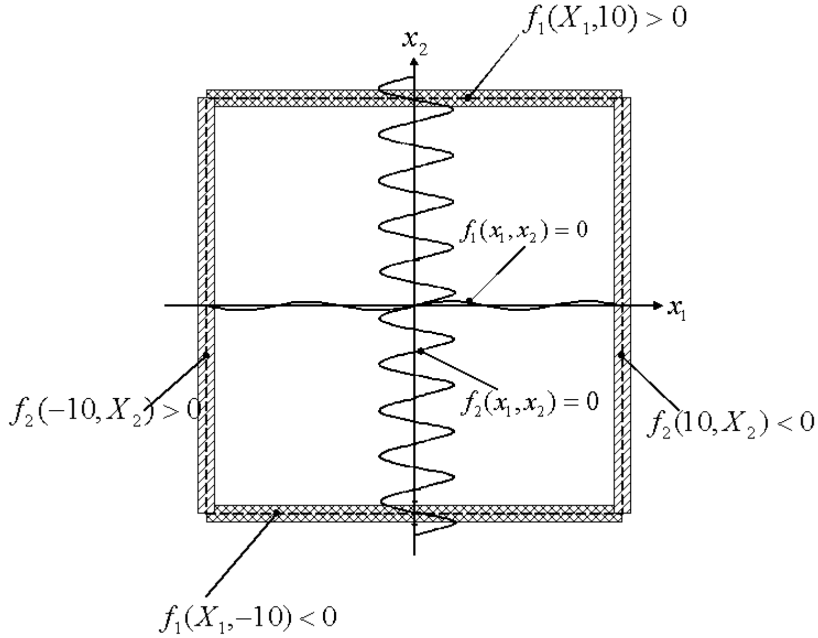

Fig. 1. Implicit graphs of $f_1(x_1, x_2) = -\sin(x_1) + 4x_2 = 0$ and $f_2(x_1, x_2) = -x_1 + \sin(2x_2) = 0$. The solution of the $n \times n$ system $F(x) = \theta$ is at the intersection of the two implicit curves.

Next example is a 1-dimensional case. In this case, the weak parametric Miranda theorem could be named the "weak parametric Intermediate Value Theorem" in this case as the Miranda theorem is reduced to the Intermediate Value Theorem in the 1-dimensional case.

Example 4. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $f(p, x) = p^2 + x^2 - 9$ and $\mathbf{p} \times \mathbf{x} = [-1, 1] \times [0, 5]$. It is easy to check of figure 2 that $\mathbf{p}$ is a subset of the projection of $f = 0$, and hence is a solution of $VPP(f = 0, \mathbf{p}, \mathbf{x})$. Now, let us use the weak parametric Miranda theorem to prove this fact :

$f(\mathbf{p}, [0, 0]) \subset [-1, 1]^2 + [0, 0]^2 - 9 = [-9, -8] < 0$ and,

$f(\mathbf{p}, [5, 5]) \subset [-1, 1]^2 + [5, 5]^2 - 9 = [16, 17] > 0$.

Eventually, the weak parametric Miranda theorem let one come to the conclusion that $S = \mathbf{p}$ is a solution of $VPP(f = 0, \mathbf{p}, \mathbf{x})$, ie :

$$\forall p \in [-1, 1], \exists x \in [0, 5], f(p, x) = 0$$

Next example introduces incertitudes on parameters of example 3.

Example 5. Let $F = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$; $F(p, x) = \begin{pmatrix} -\sin\big((1 + p_1)x_1\big) + 4x_2 \\ -x_1 + \sin\big(2(1 + p_2)x_2\big) \end{pmatrix}$ and $\mathbf{p}_1 \times \mathbf{p}_2 \times \mathbf{x}_1 \times \mathbf{x}_2 = [10^{-2}, 10^{-2}]^2 \times [-10, 10]^2$. Figure 1 shows the implicit curves

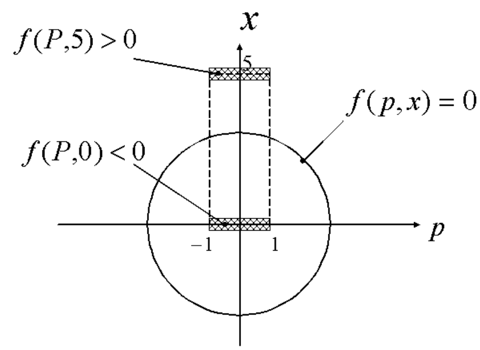

Fig. 2. Implicit graphs of $f(p, x) = p^2 + x^2 - 9$ – centered circle of radius 3

of $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ for parameters values $p_1 = 0$ and $p_2 = 0$. On can easily check that $x = (0, 0)^T$ is a solution of $F(0, 0, x) = \theta$ and, as parameters only changes the frequencies of sinusoids oscillations, this stands for every parameters values. So as to prove this, weak parametric Miranda theorem is applied in the following way - calculus are the same than in example 3 because changing the frequencies does not change the range of the involved functions :

- Following two interval evaluations : $f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_1, [-10, -10]) = [-41, -39] < 0$ and $f_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_1, [10, 10]) = [39, 41] > 0$. This let one conclude that, for every fixed parameters values, the implicit graph of $f_1(p_1, p_2, x_1, x_2) = 0$ goes continuously from $x_1 = -10$ to $x_1 = 10$ for every parameters values – the horizontal sinusoid on figure 1 for $p_1 = 0$ and $p_2 = 0$.
- Following two interval evaluations : $f_2(\mathbf{p}_1, \mathbf{p}_2, [-10, -10], \mathbf{x}_2) = [9, 11] > 0$ and $f_2(\mathbf{p}_1, \mathbf{p}_2, [10, 10], \mathbf{x}_2) = [-9, -11] < 0$. This let one conclude that, for every fixed parameters values, the implicit graph of $f_2(p_1, p_2, x_1, x_2) = 0$ goes continuously from $x_2 = -10$ to $x_2 = 10$ for every parameters values – the vertical sinusoid on figure 1 for $p_1 = 0$ and $p_2 = 0$.
- Eventually, from the four latter interval evaluations, WPM theorem let one conclude that $\forall p \in [10^{-2}, 10^{-2}]^2, \exists x \in [-10, 10] \times [-10, 10], F(p, x) = \theta$.

This section has presented the weak parametric Miranda theorem as a central tool to solve Verified Projection Problem. One can notice that many other existence theorem – for example, Brouwer theorem, Newton methods, ... — can be generalized so as to treat the parametric case. The choice of Miranda theorem relies upon its known efficiency, and upon our personal preference. Next section investigates two difficulties one is faced to implement the weak parametric Miranda theorem in the general case and propose solutions.

4 Implementation issues

In this section, we investigate to sources of instability for the application of the weak parametric Miranda theorem theorem to verified projection problem resolution : Regular systems with Jacobi matrix far from identity matrix on solution points, and splitting in variable space.

4.1 Working on the preconditioned system

It may happen that Miranda and weak parametric Miranda theorems need specific ratio between components of the box on which they are applied if the Jacobi matrix of the system is far from the identity matrix on the solution point. Figure 3 shows a simple example where this situation arises.
A common technique to overcome this problem is to precondition the system, hence obtaining a equivalent system with Jacobi matrix near the identity one – see [6] for example for the application of the Miranda theorem to preconditioned $n \times n$ systems of polynomials.
A parametric system $F(p,x) = \theta$ can be preconditioned with the mid-point matrix coming from the evaluation of the square variable part of its Jacobi matrix $\frac{\partial F}{\partial x}(p,x)$ over $\mathbf{p} \times \mathbf{x}$. In this case , the preconditioned system has a Jacoby matrix which looks like: With $M = \text{mid}\big(range_{\mathbf{p}\times\mathbf{x}}\big((\frac{\partial F}{\partial x})\big)\big)$ being invertible, as $x = \theta \Leftrightarrow M^{-1}x = \theta$, we have $F(p,x) = \theta \Leftrightarrow M^{-1}F(p,x) = \theta$. Moreover, if $\mathbf{p}$ and $\mathbf{x}$ are short enough so that F if not too far away from its tangent linear application, the interval evaluation of its Jacobi matrix $M^{-1}\frac{\partial F}{\partial x}(\mathbf{p},\mathbf{x})$ is near the identity matrix, and the weak parametric Miranda theorem will easily be applied.

Example 6. In the case of a $\mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2$ system, the preconditioned system should have a Jacobi matrix which looks like the following:

$$\begin{pmatrix} \alpha_1, 1, 0, 0 \\ \alpha_2, 0, 1, 0 \\ \alpha_3, 0, 0, 1 \end{pmatrix}$$

A problem arises when working on preconditioned systems: The expression of the new equations are linear combinations of original equations, therefore if no formal simplification can be done, many new occurrences of variables are introduced, and natural extensions are very penalized. This can be overcome by linearizing the studied system, as linear equations are easily simplified. One way to implement this idea is to change the natural extension to the centered interval form, this latter corresponding to a kind of linearization. Both following inclusions shows the expression of the centered form of a system F, and of the preconditioned system $M^{-1}F$ – with $\hat{p} \in \mathbf{p}$, $\hat{x} \in \mathbf{x}$ and M^{-1} usually approximately chose as the mid-point of the interval evaluation of the Jacoby $F'(\mathbf{p},\mathbf{x})$.

$$\underset{\mathbf{p},\mathbf{x}}{range}(F) \subset F(\hat{p},\hat{x}) + F'(\mathbf{p},\mathbf{x})(\mathbf{p}-\hat{p},\mathbf{x}-\hat{x})^T$$

$$\underset{\mathbf{p},\mathbf{x}}{range}\left(M^{-1}F\right) \subset M^{-1}F(\hat{p},\hat{x}) + M^{-1}F'(\mathbf{p},\mathbf{x})(\mathbf{p}-\hat{p},\mathbf{x}-\hat{x})^T$$

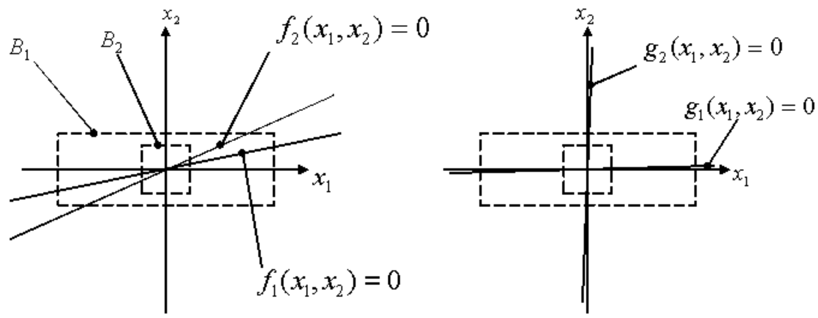

Fig. 3. Left : Initial system $F(x) = \theta$; box $\mathbf{b}_2$ has not the good ratio. Only boxes with ratio like $\mathbf{b}_1$ - x_1 component larger the x_1 one, but not too much - are adapted to Miranda theorem. Right : Preconditioned system $G(x) = M^{-1}G(x)$; same solution but both boxes are adapted to Miranda theorem.

Our next example is the same system as in example 5, but the box is chosen very closer to the solution and preconditioning is needed.

Example 7. Let F be the same function than in example 5 but $\mathbf{p}_1\times\mathbf{p}_2\times\mathbf{x}_1\times\mathbf{x}_2 = [-10^{-2}, 10^{-2}]^4$. This time, the implicit graphs of $f_1(p,x) = 0$ and $f_2(p,x) = 0$ for fixed parameters values are illustrated by figure 3 – left graph for the non-preconditioned system $F = \theta$. As $\mathbf{x}_1 \times \mathbf{x}_2$ is a square – it corresponds to box $\mathbf{b}_2$ of figure 3, weak parametric Miranda theorem cannot be applied directly and preconditioning is needed.
The Jacobi matrix of the system is the following – the two firsts columns correspond to derivatives with respect to parameters, the two last with respect to variables :

$$F' = \begin{pmatrix} -x_1\cos\big((1+p_1)x_1\big) \ , \ 0 \ , \ -(1+p_1)\cos\big((1+p_1)x_1\big) \ , \ 4 \\ 0 \ , \ 2x_2\cos\big(2(1+p_2)x_2\big) \ , \ -1 \ , \ 2(1+p_2)\cos\big(2(1+p_2)x_2\big) \end{pmatrix}$$

Its interval evaluation is included inside the following interval matrix - it is obtained by natural interval evaluation with outer rounding :

$$F'(\mathbf{p}_1,\mathbf{p}_2,\mathbf{x}_1,\mathbf{x}_2) \subset \begin{pmatrix} [-0.01,0.01] \ , \ 0 \ , \ -[0.989,1.01] \ , \ 4 \\ 0 \ , \ [-0.02,0.02] \ , \ -1 \ , \ [1.978,2.02] \end{pmatrix}$$

We want the last two columns to be changed for a near identity matrix by the preconditioning. Therefore, the preconditioning matrix M^{-1} is chosen as the approximate mid-point inverse of the last two columns :

$$M^{-1} \approx \begin{pmatrix} 1 \ , \ 4 \\ -1 \ , \ 2 \end{pmatrix}^{-1} \approx \begin{pmatrix} 1 \ , \ -2 \\ 0.5 \ , \ -0.5 \end{pmatrix}$$

Now, we study the preconditioned system $G = M^{-1}F$ which has exactly the same solutions than the original one, as M^{-1} is invertible.

$$G = M^{-1}F = \begin{pmatrix} 1 \times f_1(p_1,p_2,x_1,x_2) + (-2) \times f_2(p_1,p_2,x_1,x_2) \\ 0.5 \times f_1(p_1,p_2,x_1,x_2) + (-0.5) \times f_2(p_1,p_2,x_1,x_2) \end{pmatrix}$$

Moreover, the variable part of its Jacobi has a near identity matrix for every parameters values : A direct formal check of the product let one conclude that $G' = M^{-1}F'$. An interval evaluation of the Jacoby of G is then easily obtained by $G'(\mathbf{p},\mathbf{x}) \subset M^{-1}F'(\mathbf{p},\mathbf{x})$:

$$G'(\mathbf{p},\mathbf{x}) = \begin{pmatrix} [-0.01, 0.01]\ ,\ [-0.04, 0.04]\ ,\ [0.99, 1.012]\ ,\ [-0.041, 0.045] \\ [-0.005, 0.005]\ ,\ [-0.01, 0.01]\ ,\ [-0.006, 0.006]\ ,\ [0.99, 1.012] \end{pmatrix}$$

We can now apply the weak parametric Miranda theorem to the preconditioned system. Unfortunately, the preconditioning introduced many new occurrences, and the natural extension cannot conclude. In such a situation, as preconditioning needs a box where the function F is not too far from its tangent linear application, centered form is going to be very efficient[1] :

$$\begin{aligned} g_1(\mathbf{p}_1, \mathbf{p}_2, 0.01, \mathbf{x}_2) \subset\ & g_1(0, 0, 0.01, 0) \\ & + (\mathbf{p}_1 - 0) \times \big(G'(\mathbf{p},\mathbf{x})\big)_{1,1} \\ & + (\mathbf{p}_2 - 0) \times \big(G'(\mathbf{p},\mathbf{x})\big)_{1,2} \\ & + ([0.01, 0.01] - 0.01) \times \big(G'(\mathbf{p},\mathbf{x})\big)_{1,3} \\ & + (\mathbf{x}_2 - 0) \times \big(G'(\mathbf{p},\mathbf{x})\big)_{1,4} \\ & \subset [0.00905, 0.01096] > 0 \end{aligned}$$

In the same way, one obtains :

$$g_1(\mathbf{p}_1, \mathbf{p}_2, -0.01, \mathbf{x}_2) \subset [-0.01096, -0.00905] < 0,$$
$$g_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_1, 0.01) \subset [0.00934, 0.01067] > 0,$$
$$g_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{x}_1, -0.01) \subset [-0.01067, -0.00934] < 0,$$

This let one conclude through weak parametric Miranda theorem that $\forall p \in \mathbf{p}, \exists x \in \mathbf{x}, G(p,x) = \theta$. Which is equivalent ton $\forall p \in \mathbf{p}, \exists x \in \mathbf{x}, F(p,x) = \theta$.

Linear systems with quantified interval incertitudes have been a lot studied, see for examples [15] and [14]. In the methods proposed in those papers, dependencies between coefficients cannot be handled. These dependencies arises for example if one knows that involved matrices are symmetric - see [1]. Next examples illustrates that the method we propose in this paper can handle this kind quantified systems. Moreover, as these systems are linear, calculus are rather simple and multi-incidences introduced by preconditioning can be handle with natural extensions as formal simplification are very easy. The studied quantified linear system can be changed to a parametric function and a verified projection problem. Next example illustrates this transformation.

Remark 5. Up to now, we only presented a method for dealing with $n \times n$ parametric systems. Method to deal with under-constrained $m \times n$ systems, $m < n$,

[1] Interval evaluations are used to evaluate the range of the functions of the faces of the studied box. Hence, interval evaluation of the Jacobi matrix can be done on this restricted sets, but in our case, the its evaluation on the whole box is enough to conclude. It should be the enough in general as, so as to be preconditioned, the system has to be near its tangent linear application, and hence derivatives do not change a lot inside the box.

are proposed in conclusion. Quantified linear system of next example have been chosen so that the resulting parametric system to be used together with weak parametric Miranda theorem is square. Examples 2 and 3 of [1] are cases where the resulting parametric function is not a square system. On the other hand, example 5 of [1] illustrates a case where the resulting parametric system is over-constrained, leading to a null volume solution set.

Example 8. Let us study the following symmetric linear system :

$$S_{a,b,c}(x_1,x_2) \Leftrightarrow \begin{pmatrix} 100x_1 + (a-100)x_2 & = 2000 + c \\ (a-100)x_1 + (25+b)x_2 & = 625 + c \end{pmatrix},$$

$a \in \mathbf{a} = [-10,10]$, $b \in \mathbf{b} = [-10,10]$, $c \in \mathbf{c} = [-1,1]$.

Incertitudes of coefficient are not independent : One can see what ever are the values of a, b and c, all corresponding systems are symmetric. Incertitudes are quantified as following : We look for the generalized solution set $\Xi = \{x \in \mathbb{R}^2 | \forall c \in \mathbf{c}, \exists a \in \mathbf{a}, \exists b \in \mathbf{b}, S_{a,b,c}(x)\}$.

The following question, "is $\mathbf{x} = [14,16] \times [34,36]$ [2] a subset of Ξ ?" can be easily viewed as a verified projection problem: $\forall (c,x_1,x_2) \in \mathbf{c} \times \mathbf{x}_1 \times \mathbf{x}_2, \exists (a,b) \in \mathbf{a} \times \mathbf{b}, F(c,x_1,x_2,a,b) = \theta$, with the following function:

$$F(c,x_1,x_2,a,b) = \begin{pmatrix} 100x_1 + (a-100)x_2 - 2000 + c \\ (a-100)x_1 + (25+b)x_2 - 625 + c \end{pmatrix}$$

$S_{a,b,c}(x_1,x_2) \Leftrightarrow F(c,x_1,x_2,a,b) = \theta$ and c, x_1 and x_2 considered as parameters and a and b as variables due to the quantification of the solution set.

Let us precondition this system so as to apply the weak parametric Miranda theorem . Variables are ordered as above for the expression of the Jacobi of the system.

$$F' = \begin{pmatrix} 1 \ , \ 100 \ , \ -100 \ , \ x_2 \ , \ 0 \\ 1 \ , \ -100 \ , \ 25 \ , \ x_1 \ , \ x_2 \end{pmatrix}$$

The approximate mid-point inverse of the interval evaluation of its last two columns is the following:

$$M^{-1} = \begin{pmatrix} 35 \ , \ 0 \\ 15 \ , \ 35 \end{pmatrix}^{-1} \approx \begin{pmatrix} 0.0280 \ , \ 0 \\ -0.0140 \ , \ 0.0280 \end{pmatrix} \approx (140)^{-1} \begin{pmatrix} 2 \ , \ 0 \\ -1 \ , \ 2 \end{pmatrix}.$$

We can now build the preconditioned system $G = 140M^{-1}F$ – the constant has no repercussion on the application of the weak parametric Miranda theorem and simplifies all calculus. Here, the simplicity of the original linear system let us cancel multi-occurrences from the expression of G.

$$G = \begin{pmatrix} g_1(c,x_1,x_2,a,b) \\ g_2(c,x_2,x_2,a,b) \end{pmatrix} = \begin{pmatrix} 4000 + 2c + 200x_1 - 200x_2 + 2x_2a \\ -750 + c - 300x_1 + 150x_2 + (2x_1 - x_2)a + 2x_2b \end{pmatrix}$$

Eventually, followings four interval evaluations let one apply weak parametric Miranda theorem and positively answer to the verified projection problem and hence, to the question of this example : $\mathbf{x} \subset \Xi$.

- $g_1(\mathbf{c},\mathbf{x}_1,\mathbf{x}_2,+10,\mathbf{b}) = [278,1122] > 0$

[2] For $a=0$, $b=0$ and $c=0$, the solution of the system is $x_1 = 15$ and $x_2 = 35$.

- $g_1(\mathbf{c}, \mathbf{x}_1, \mathbf{x}_2, -10, \mathbf{b}) = [-1122, -278] < 0$
- $g_2(\mathbf{c}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{a}, +10) = [147, 1253] < 0$
- $g_2(\mathbf{c}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{a}, -10) = [-1253, -147] > 0$

4.2 Using contractor for splitting stability

As mentioned in [11] in paragraph 6 - Save Branching - splitting a variable component of a box can lead to unstable sub-constraints. In [11], the author chose not to split this variables. In our situation, they will have to be split. Therefore, we have to design a safe method.
To this end, let us study a situation where splitting leads to unstable sub-constraints. Let us remind the system of example 4 : $f(p, x) = p^2 + x^2 - 9$. Let us change the studied box so that only a part of the box is inside the projection : $\mathbf{p} \times \mathbf{x} = [1, 4] \times [0, 5]$. This new situation is illustrated on figure 4 at upper-left graph.

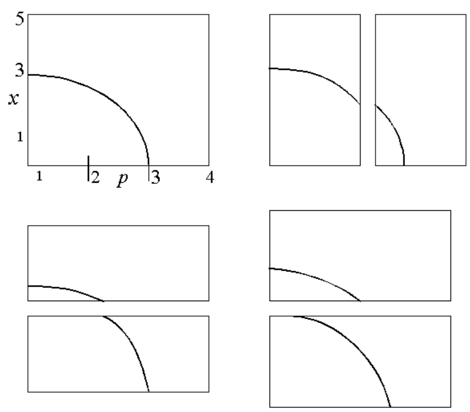

Fig. 4. Implicit graphs of $f(p, x) = p^2 + x^2 - 9$ - centered circle of radius 3

On one hand, if one splits the parameter component of the box - upper-left graph - one obtains two boxes, one inside the projection where WPM theorem applies ; and the other one looking like the initial box. No instability is introduced.
On the other hand, if one splits the variable component of the box at $x = 2.5$, one obtains the two boxes on the bottom-left graph. This time, the corresponding parameter $p = \sqrt{9 - 2.5^2} \approx 1.65$ became an unstable parameter and we will not be any more able to prove it is inside the projection. This can be avoided by a "safe split" : The variable component is split keeping a short interval overlapping the two components – bottom-right graph : $[1, 4] \to [1, 2.6] \cup [2.4, 4]$. Unfortunately, this technique will lead to a clustering effect around $p = 2.5$: Many small boxes will appear near $p = 1.65$ during the bisection process.

A better solution would be to design a method which would answer the subset of the parameter box which is included in the projection. This is done by changing weak parametric Miranda theorem to a system of constraints in the parameter space - see next two examples for the transformation details.
This system is made of inequalities on parameters, ie $\{c_1(p) < 0, ..., c_n(p) < 0\}$. We can then extract a subspace of P which contains only solutions following the method proposed in [4][3]: We apply a contractor which looses no solutions to negation of each constraints independently, i.e. $c_k(p) \geq 0$, and the p-values discarded inside every constraints negation contraction are proved to verify all the $c_k(p) < 0$.
Next two examples illustrates the proposed approach.

Example 9. A 1-dimensional example - figure 4.
Let $f(p, x) = p^2 + x^2 - 9$ and $\mathbf{p} \times \mathbf{x} = [1, 4] \times [0, 5]$ define a verified projection problem. Then, weak parametric Miranda theorem leads to the following sufficient condition for a parameter value to be a solution:

$$[f(p, 0) \geq 0 \wedge f(p, 5) \leq 0] \vee [f(p, 0) \leq 0 \wedge f(p, 5) \geq 0].$$

Solutions of the constraint $c_1(p) : f(p, 0) * f(p, 5) < 0$ [4] inside $\mathbf{p}$ are solutions of $VPP(f = 0, \mathbf{p}, \mathbf{x})$.
Now, following [4] we inject the negation $\big(f(p, 0) * f(p, 5) \geq 0\ ,\ \mathbf{p}\big)$ in a contractor. The p-values discarded are then proved to be solutions of the initial verified projection problem. We implemented this using 3B-consistency - [9] - and had the following answer :

$\neg c_1(p)$ contracts $[1, 4]$ to $[2.9999999999999987, 4]$

Eventually, one come to the conclusion that $S = [1, 2.9999999999999986]$ is a solution to $VPP(f = 0, \mathbf{p}, \mathbf{x})$. No splitting is needed.

Example 10. A 2-dimensional example - intersection of a sphere and a plan[5].
Let $F(p, x) = \begin{pmatrix} f_1(p, x_1, x_2) \\ f_2(p, x_1, x_2) \end{pmatrix} = \begin{pmatrix} p^2 + x_1^2 + x_2^2 - 9 \\ p + x_1 + 100x_2 \end{pmatrix}$ and $\mathbf{p} \times \mathbf{x}_1 \times \mathbf{x}_2 = [1, 4] \times [0, 5] \times [-1, 1]$ define a verified projection problem.
The constraint system implying a validated application of weak parametric Miranda theorem is the following :

$c_1(p) : f_1(p, 0, \mathbf{x}_2) * f_1(p, 5, \mathbf{x}_2) < 0$

$c_2(p) : f_2(p, \mathbf{x}_1, -1) * f_1(p, \mathbf{x}_1, 1) < 0$

Then, independent contractions of the negation of the constraints lead to :

$\neg c_1(p)$ contracts $[1, 4]$ to $[2.8284271247461885, 4]$

$\neg c_2(p)$ contracts $[1, 4]$ to $\emptyset$

Eventually, $S = [1, 2.8284271247461884]$ is proved to be a solution of .

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3 In fact, our situation is nearer to the one described in [3], this latter being similar to [4]

4 Strict inequality is needed to prevent special cases, but they do not affect the efficiency of the method.

5 The plan is chosen hardly horizontal so that we do not need preconditioning in this example. Implementation of a contractor which uses centered form for every variables is yet done.

In the last example, the contractions of the negations of the constraints do not lead to the optimal solution. Splitting and pruning have to be used to this end. This method allows splitting the variables components without any clustering effects.

5 About regularity of solutions

In the case, of next example, no numerical method can raise some parameter values which are inside the projection. This is due to the null derivative $\frac{\partial f}{\partial x}$. This is generalized to higher dimension by considering the quantity $\det\left(\frac{\partial F}{\partial x}\right)$. This section shows that this latter quantity can be used to design a sufficient condition for a parameters value numerically proved to belong to the wanted projection.

Example 11. Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$; $f(p, x) = (p - x)^2$ and $\mathbf{p} \times \mathbf{x} =]-\infty, +\infty[^2$. In this case, $S =]-\infty, +\infty[$ is a solution of $VPP(f = 0, \mathbf{p}, \mathbf{x})$ because $\forall p \in S, \exists x \in \mathbf{x}_{(x=p)}, f(p, x) = 0$. In spite of that, no p can be shown to be inside the projection through numerical methods as the problem is ill-posed for every parameter values.

In order to give a sufficient condition for the possibility to show through numerical methods that a parameter value is inside the projection, the following definition is proposed.

Definition 2 (Regular parameter value).
Let $F : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be differentiable inside a given box $\mathbf{p} \times \mathbf{x} \subset \mathbb{R}^p \times \mathbb{R}^n$. A regular parameter value p inside $\mathbf{p} \times \mathbf{x}$ is a parameter value for which there exists a solution $x \in \mathbf{x}$ to the system $F(p, x) = \theta$ where the variable part of Jacobi matrix $\frac{\partial F}{\partial x}$ is invertible, i.e. :

p is a regular parameter value inside $p \times \mathbf{x}$ if and only if $\exists x \in \mathbf{x}, \big(F(p, x) = \theta \wedge \det\big(\frac{\partial F}{\partial x}(x)\big) \neq 0\big)$

Next proposition is the main property about the possibility to show that a regular parameter value is inside the projection of the system solution set.

Proposition 1. *If p^* is a regular parameter value of $F : D_F \subset \mathbb{R}^{p+n} \rightarrow \mathbb{R}^n$ inside $\mathbf{p} \times \mathbf{x} \subset D_F$, then their exists a small box $\mathbf{b} \subset \mathbf{p} \times \mathbf{x}$, $p^* \in \mathbf{b}$, such that weak parametric Miranda theorem applies on the preconditioned system.*

Proof. The proof is presented in the case $p = 1$ and $n = 2$. Generalization is strait forward. For reading convenience, we will denote the interval – or a box if a and b are real vectors, inequalities being considered component wise – $[a - b, a + b]$, $b \geq 0$, by $a \pm b$.
As p^* is a regular parameter, $\exists x^* \in \mathbf{x}$ such that $F(p^*, x^*) = \theta$ and if $M = (\frac{\partial F}{\partial x})(p^*, x^*) \in \mathbb{R}^{2\times 2}$, then $\det(M) \neq 0$.
Define $D = M^{-1}F'(p^*, x^*)$. It is easy to check that D has the following form, α and β being real numbers.

$$D = \begin{pmatrix} \alpha \ , \ 1 \ , \ 0 \\ \beta \ , \ 0 \ , \ 1 \end{pmatrix}$$

Given a positive real ϵ, we then build the the following interval matrix:

$$\mathbf{D}_\epsilon = \begin{pmatrix} \alpha \pm \epsilon \ , \ 1 \pm \epsilon \ , \ 0 \pm \epsilon \\ \beta \pm \epsilon \ , \ 0 \pm \epsilon \ , \ 1 \pm \epsilon \end{pmatrix}$$

We are now in position to consider the preconditioned system $G = M^{-1}F$, $G = (g_1, g_2)^T$. Its Jacoby matrix is $G' = M^{-1}F$. It is the easy to check that $G'(p^*, x^*) = D$. As all derivatives are continuous and interval extensions are considered as continuous, one can make $G'(\mathbf{b})$ arbitrary close to D by considering for example boxes $\mathbf{b}_{w_p,w_x} := (p^*, x_1^*, x_2^*) \pm (w_p, w_x, w_x)$ – i.e. $[p^* - w_p, p^* + w_p] \times [x_1^* - w_x, x_1^* + w_x] \times [x_2^* - w_x, x_2^* + w_x]$ – which width tend to 0 when (w_p, w_x) tends to $(0, 0)$. This can be formalized through the following proposition:

$$\forall \epsilon, \exists (w_p, w_x) \text{ s.t. } G'(\mathbf{b}_{w_p,w_x}) \subset \mathbf{D}_\epsilon$$

Furthermore, as monotonic interval extensions are considered, the inclusion stays valid if a smaller box $\mathbf{b}_{w'_p,w'_x}$ is considered, leading to the following proposition:

$$\forall \epsilon, \exists (w_p, w_x) \text{ s.t. } (w'_p, w'_x) \leq (w_p, w_x) \Rightarrow G'(\mathbf{b}_{w'_p,w'_x}) \subset \mathbf{D}_\epsilon$$

Then, consider (w_p, w_x) such that $\mathbf{b}_{w_p,w_x} \subset \mathbf{p} \times \mathbf{x}$ and that previous proposition is satisfied for $\epsilon \leq 0.01$ – the interval evaluation of Jacoby will be strongly regular. We now check the hypothesis of the weak parametric Miranda theorem using centered form on the preconditioned system, and will come to the conclusion that one can decrease w_p so that the weak parametric Miranda theorem applies. Let us see an example with one face of the considered box, $\mathbf{b}_{1+} := (p^*, x_1^* + w_x, x_2^*) \pm (w_p, 0, w_x)$ – i.e. $[p^* - w_p, p^* + w_p] \times [x_1^* + w_x, x_1^* + w_x] \times [x_2^* - w_x, x_2^* + w_x]$:

$$g_1(\mathbf{b}_{1+}) \subset g_1(p^*, x_1^* + w_x, x_2^*) + \frac{\partial g_1}{\partial p}(\mathbf{b}_{1+}) \times (0 \pm w_p) + \frac{\partial g_1}{\partial x_2}(\mathbf{b}_{1+}) \times (0 \pm w_x)$$

Which implies, as $G'(\mathbf{b}_{+1}) \subset G'(\mathbf{b}_{w_p,w_x,w_x}) \subset D_\epsilon$,

$$g_1(\mathbf{b}_{1+}) \subset g_1(p^*, x_1^* + w_x, x_2^*) + (\alpha \pm \epsilon) \times (0 \pm w_p) + (0 \pm \epsilon) \times (0 \pm w_x)$$

It remains to evaluate $g_1(p^*, x_1^* + w_x, x_2^*)$ using once more the centered form:

$$g_1(p^*, x_1^* + w_x, x_2^*) \in g_1(p^*, x_1^*, x_2^*) + \frac{\partial g_1}{\partial x_1}(\mathbf{b}) \times (w_x) \subset (1 \pm \epsilon) \times w_x$$

Putting all things together, one obtains:

$$g_1(\mathbf{b}_{1+}) \subset (1 \pm \epsilon) \times w_x + (\alpha \pm \epsilon) \times (0 \pm w_p) + (0 \pm \epsilon) \times (0 \pm w_x)$$

One obtains the following expression by interval arithmetic:

$$g_1(\mathbf{b}_{1+}) \subset \Big(w_x \pm \big(2\epsilon w_x + (\alpha + \epsilon)\, w_p \big) \Big) = w_x \left(1 \pm \left(2\epsilon + (\alpha + \epsilon) \frac{w_p}{w_x} \right) \right)$$

It is now easy to see that on can decreasing w_p to $w'_p < w_p$ making this expression strictly positive. The same can be done with the other faces, and eventually taking the minimum of the 4 w'_p calculated leads to a positive application of the weak parametric Miranda theorem . □

Remark 6. This proposition is somewhat theoretical, as the proof uses boxes centered of the true solutions. Transposing it to a real branch and prune algorithm would need to take into account the fact that, in the best case, using local search, the box can be centered on an approximate 0 of the function.

This latter proposition does not take into account the finite precision of the calculus. Next three examples show different one dimensional example, and focuses on parameter values which are not regular.

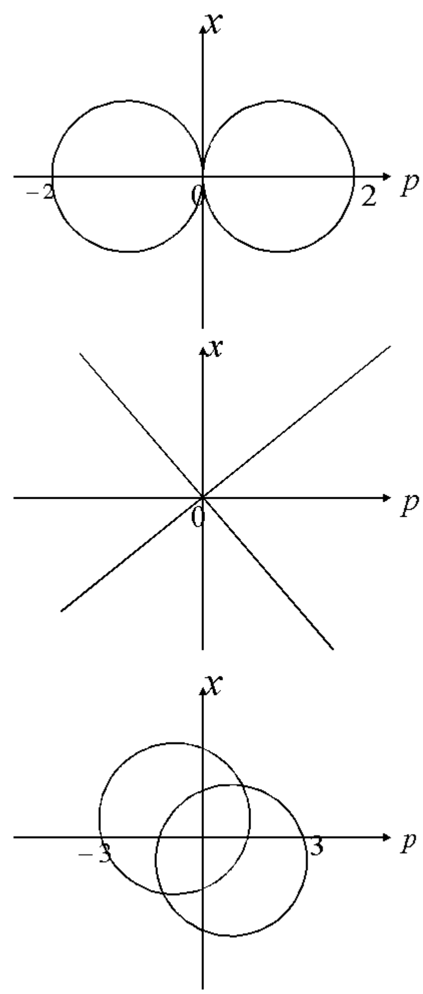

Fig. 5. Up : $p \in \{-2, 0, 2\}$ are not regular parameter values ; Middle : $p = 0$ is not a regular parameter value ; Down : $p \in \{-3, 3\}$ are not regular parameter values.

Example 12. Figure 5, up.
Let $f(p,x) = \big((p-1)^2+x^2-1\big)\big((p+1)^2+x^2-1\big)$. For $p \in \{-2,0,2\}$, the solution is $x = 0$ and $\det(F(p,.)'(x)) = \left|\frac{\partial f_p}{\partial x}(x)\right| = 0$, hence these parameter values are not regular parameter values.

Example 13. Figure 5, middle.
Let $f(p,x) = (p-x)(p+x)$. For $p = 0$, the solution is $x = 0$ and $\det(F(p,.)'(x)) = \left|\frac{\partial f_p}{\partial x}(x)\right| = 0$, hence this parameter values is not regular parameter values.

Example 14. Figure 5, down.
Let $f(p,x) = \big((p-1)^2+(x-1)^2-4\big)\big((p+1)^2+(x+1)^2-4\big)$. This time, excepted for the boundary of the projection, for each parameter value in the projection one can find a solution for which $\det(F(p,.)'(x)) = \left|\frac{\partial f_p}{\partial x}(x)\right| \neq 0$.

6 Conclusion

In this paper, we presented the theoretical basis for solving the Verified Projection Problem. This relies upon a weak version of parametric Miranda theorem. We showed that preconditioning the system was an important step. Centered form is also an important step as preconditioned systems are highly more accurately bounded with this latter than with natural interval extension. We also designed a quantifier free inequality constraints system which solutions are proved by WPM theorem to be inside the projection. This last step increases the stability of splitting variables components of a box. Eventually, we showed that if a parameter value is regular, one can always find a little box around it where weak parametric Miranda theorem applies, showing that this parameter value is inside the wanted projection.
In order to build a complete branch and bound algorithm, we will have to design a specific constraint solver which will work on centered form for every variables. Then benchmarks will be tested - including examples 2.2 and 4.1 of [8].
A future topic of research will be to extend this method to under-constrained systems : A way to explore is to change a $n \times m$ system - $n < m$ - to a $m \times m$ one, adding linear equations which will be locally chosen so as to make the interval evaluation of Jacobi matrix of the $m \times m$ system the most regular as possible.
An other way to deal with under-constrained systems are Modal Interval - [16]. n-dimensional interpretation of modal rational extensions have almost the same proving power than WPM theorem, but are not restricted to $n \times n$ systems. For example, in 1-dimensional cases like example 4, a simple modal rational extension evaluation leads to the same answer than the WPM theorem : $f^*([-1,1],[5,0]) = [-1,1]^{2*} +^* [5,0]^{2*} -^* 9 = [16,-8]$, which is interpreted as $\forall p \in [-1,1], \forall z \in [-8,16], \exists x \in [0,5], z = f(p,x)$, which eventually implies $\forall p \in [-1,1], \exists x \in [0,5], f(p,x) = 0$.

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