Automatica 100 (2019) 371–377

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Estimating the robust domain of attraction for non-smooth systems using an interval Lyapunov equation☆

Alexandre Goldsztejn[a,*], Gilles Chabert[b]

[a] CNRS, LS2N, France
[b] IMT Atlantique, LS2N, France

ARTICLE INFO

Article history:
Received 31 January 2017
Received in revised form 29 September 2017
Accepted 31 January 2018

Keywords:
Nonlinear systems
Exponentially stable fixed point
Domain of attraction
Lyapunov function
Interval analysis

ABSTRACT

The Lyapunov equation allows finding a quadratic Lyapunov function for an asymptotically stable fixed point of a linear system. Applying this equation to the linearization of a nonlinear system can also prove the exponential stability of its fixed points. This paper proposes an interval version of the Lyapunov equation, which allows investigating a given Lyapunov candidate function for non-smooth nonlinear systems inside an explicitly given neighborhood, leading to rigorous estimates of the domain of attraction (EDA) of exponentially stable fixed points. These results are developed in the context of uncertain systems. Experiments are presented, which show the interest of the approach including with respect to usual approaches based on sum-of-squares for the computation of EDA.

1. Introduction

This paper addresses the computation of estimates of the domain of attraction (EDA), i.e., subsets of the domain of attraction (DA), of exponentially stable fixed points for nonsmooth uncertain systems. It is well known that the size of the DA is not related to the strength of attraction of the fixed point, as shown by the following simple example: the system $x'(t) = f(x(t)) = \theta^3 x(t)^3 - \theta x(t)$ with $\theta > 0$, for which the origin is an attracting fixed point. Since $f'(0) = -\theta$, the greater θ the stronger the attraction of the origin. However, the DA of 0 is the open interval $(-\theta^{-1}, \theta^{-1})$, which decreases as θ increases. The computation of EDA hence brings complementary information to the study of the system linearization at the fixed point, and is of practical importance.

Lyapunov functions have turned out to be a central tool in the study of the stability of fixed points. They can also be used to estimate their DA since, under mild assumptions, level sets of Lyapunov functions are included inside the DA. Proving that a function v is a Lyapunov function involves proving that a related function, namely the Lie derivative $\dot{v}(x) = \nabla v(x)^T f(x)$ of the Lyapunov function with respect to the flow, is negative definite (ND), i.e., negative inside a given neighborhood of the fixed point except at the fixed point where it is zero. This can be done formally for simple systems. However, numerical algorithms, which are required for more realistic problems, are faced to a specific difficulty for proving this property: Indeed, $\dot{v}(x)$ gets arbitrarily close to zero as x approaches the fixed point, hence preventing any attempt from proving directly numerically that it is negative except at the fixed point. This issue has been resolved in the framework of two different theories by checking indirectly that $\dot{v}$ is ND.

First, in the context of polynomial dynamical systems, proving that the Lie derivative is ND inside a region can be expressed as finding positive definite (PD) polynomials satisfying some constraints. Finding such PD polynomials can be relaxed to finding polynomials that are sum of squares (SOS). Finding such SOS polynomials amounts to solve linear matrix inequalities (LMIs), LMIs being convex and solved by efficient solvers today. This way, SOS polynomials have been applied to prove that functions are Lyapunov functions for polynomial systems, see, e.g., Chesi (2011a), Papachristodoulou and Prajna (2002) and Tibken (2000), and further extended to non-polynomial systems (Chesi, 2009; Papachristodoulou & Prajna, 2005; Wu, Yang, & Lin, 2014) and to hybrid systems (Luk & Chesi, 2015). In spite of its powerfulness, this approach presents several drawbacks: Its computational complexity turns out to be sensitive to both the dimension of the system and the degree of the polynomials involved in the description of the system (see Section 5.6). Furthermore, the current developments of this approach do not allow to handle fixed points that depend on the uncertainties. Finally, this approach

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Luca Zaccarian under the direction of Editor Daniel Liberzon.

* Correspondence to: ECN, 1 rue de la Noë, 44300 Nantes, France .
E-mail addresses: alexandre.goldsztejn@ls2n.com (A. Goldsztejn), gilles.chabert@imt-atlantique.fr (G. Chabert).

https://doi.org/10.1016/j.automatica.2018.03.036
0005-1098/

happens to lack rigor in the computed results: LMI solvers are not certified, e.g., a relatively small uncertainty on the fixed point can cause the warning "SeDuMi numerical problems warning" of the LMI solver SeDuMi used in SMRSOFT (Chesi, 2011b), which is difficult to interpret. Verified automatic procedures for LMI error bounds (Jansson, Chaykin, & Keil, 2008) could be used, but were not investigated in this context up to our knowledge.

Second, interval analysis can be used to check the sign of a given function, either by simple interval evaluation or using branch and bound algorithms. However, this approach fails because of the issue mentioned above. A more elaborated application of interval analysis to this end was proposed in Delanoue, Jaulin, and Cottenceau (2015): The Lie derivative is formally computed and its Hessian is evaluated over a box centered on the fixed point. The negative definiteness of this interval matrix, which can be checked automatically, then implies that the Lie derivative is concave and therefore negative (provided that in addition it is zero at its maximum), hence that the Lyapunov candidate function is actually a Lyapunov function. This test is called the *interval Hessian test* in the rest of the paper.

We also mention here (Giesl & Hafstein, 2014), which builds rigorously continuous piecewise affine Lyapunov functions by using an upper bound on the second derivatives, and in the context of saturated linear systems (Hu, Goebel, Teel, & Lin, 2005), which proposes to combine several quadratic functions to build larger EDA, e.g., under the form of the convex hull of ellipsoids. Finally, although not directed related to the present work, Ratschan and She (2010) proposed an interval method that builds a Lyapunov-like function to prove some capture property in a more general setting than fixed point stability. However, it requires an initial EDA.

In the present paper, we propose another approach based on interval analysis: So-called slope matrices are used to rigorously linearize the system and the Lyapunov equation dedicated to linear systems is applied to this linearization, leading to a sufficient condition that requires only the interval evaluation of the first derivatives of the flow. This approach is carefully developed so as to handle uncertain dynamical systems, hence computing estimates of the robust domain of attraction (ERDA), and non-quadratic Lyapunov functions. Finally, we define a nonlinear optimization problem that allows enlarging this initial ERDA in an optimal sense defined below. We stress that the present paper considers a given Lyapunov candidate function and addresses the problem of proving that it is actually a Lyapunov function and building ERDA from it. The problem of finding good Lyapunov function candidates is not in the scope of this paper.

The paper is organized as follows: Section 2 introduces the basics of interval analysis that will be used. Section 3 presents the interval Lyapunov equation (4), which provides a verifiable sufficient condition for a Lyapunov candidate function to actually be Lyapunov function. Section 4 shows how to use the interval Lyapunov equation to build an ERDA, as well as how to extend this initial ERDA by solving a nonlinear optimization problem. Finally, Section 5 presents several academic systems to evaluate the proposed approach and compares it to the state of the art.

2. Interval analysis

Interval analysis is a branch of numerical analysis that was born in the 1960's. It consists in computing with intervals of reals instead of reals, providing a framework for handling uncertainties and verified computations (see e.g. Jaulin, Kieffer, Didrit, and Walter (2001) and Neumaier (1990)). The reader is assumed to be familiar with interval analysis. Nevertheless, details about slope matrices are given below. Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, a box $[x] \in \mathbb{IR}^n$ and a vector $\tilde{x} \in [x]$, a $(f, [x], \tilde{x})$-slope matrix enclosure is an interval matrix $[S]$ such that

$$\forall x \in [x], \exists S(x) \in [S], f(x) = f(\tilde{x}) + S(x)(x - \tilde{x}). \quad (1)$$

The matrix $S(x)$ is called a $(f, [x], \tilde{x})$-slope matrix and $[S]$ its enclosure. For differentiable functions f, a slope matrix enclosure can be computed using interval evaluations of the derivatives: $[S] = [Df]([x])$ is a $(f, [x], \tilde{x})$-slope matrix enclosure, which actually holds for any $\tilde{x} \in [x]$. Hansen proposed in Hansen (1969) to evaluate the derivatives in a more clever way on sub-boxes of $[x]$ to obtain a tighter $(f, [x], \tilde{x})$-slope matrix enclosure:

$$[S]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]([x_1], \ldots, [x_j], \tilde{x}_{j+1}, \ldots, \tilde{x}_n). \quad (2)$$

Hansen's matrix is sharper than the simple interval evaluation of the derivatives, but is valid only for a fixed $\tilde{x}$.

Both slope matrix enclosures can be generalized to non-differentiable Lipschitz functions f using generalized gradients (Clarke, 1990) and the corresponding mean-value theorem, or the inclusion algebra described in Section 2.3 of Neumaier (1990). Although not detailed here, slope matrices can also be computed using a slope arithmetic (Rump, 1996). This approach can lead to tighter slope matrix enclosure, in particular when max functions are involved, e.g., for systems with saturations (see Appendix for an example of INTLAB (Rump, 1999) code that computes a slope matrix enclosure using the slope arithmetic).

3. Verification of Lyapunov functions

In this section, we consider a Lipschitz continuous vector field $f : [u] \subseteq \mathbb{R}^n \to \mathbb{R}^n$, where $[u]$ is a bounded neighborhood of the origin, such that $f(0) = 0$. We also consider a twice differentiable Lyapunov candidate function $v : [u] \to \mathbb{R}$, i.e., $v(0) = 0$ and v is positive definite (PD) inside $[u]$. We say that this Lyapunov candidate function is a $[u]$-Lyapunov function if furthermore $\forall u \in [u]\backslash\{0\}, \dot{v}(u) < 0$, where $\dot{v}(u) = \nabla v(u)^T f(u)$. From Lyapunov theory, the existence of such a Lyapunov function entails that the origin is asymptotically stable, and that any level set $\{u \in \mathbb{R}^n : v(u) \le c\} \subseteq [u]$ is included inside the DA of the origin.

We consider a $(f, [u], 0)$-slope matrix $A(u) \in [A]$, i.e., $\forall u \in [u], \exists A(u) \in [A], f(u) = A(u)u$, and a $(\nabla v, [u], 0)$-slope matrix $P(u) \in [P]$, i.e., $\forall u \in [u], \exists P(u) \in [P], \nabla v(u) = P(u)u$. Note that neither $P(u)$ nor $[P]$ are symmetric in general. These slope matrices allow rewriting the Lie derivative of v under the following form:

$$\dot{v}(u) = \nabla v(u)^T f(u) = u^T P(u)^T A(u)\, u. \quad (3)$$

The following proposition uses this quadratic-like expression of the Lie derivative (3) to extend the Lyapunov equation for linear systems to the nonlinear system in the neighborhood $[u]$.

Proposition 1. *Given a Lyapunov candidate function v, define $Q(u) = P(u)^T A(u) + A(u)^T P(u)$ and suppose that $Q([u])$ contains only symmetric negative definite (SND) matrices. Then v is a $[u]$-Lyapunov function for f. In particular, v is a $[u]$-Lyapunov function if the interval matrix*

$$[Q] = [P]^T [A] + [A]^T [P] \quad (4)$$

is SND, i.e., all symmetric matrices inside $[Q]$ are SND.

Proof. The same quadratic-like form as (3) is obtained by symmetrizing its matrix, i.e., $\dot{v}(u) = \frac{1}{2} u^T \big(P(u)^T A(u) + A(u)^T P(u)\big) u$, which is equal to $\frac{1}{2} u^T Q(u) u$ by definition of $Q(u)$. Since $Q(u)$ is assumed to be SND, $\dot{v}(u) < 0$ for $u \in [u] \setminus \{0\}$, and v is proved to be a $[u]$-Lyapunov function for f. Finally, since $P(u) \in [P]$ and $A(u) \in [A]$, we have $Q(u) \in [Q]$. Therefore, $[Q]$ SND implies $Q(u)$ SND for all $u \in [u]$, and the first statement implies that v is a $[u]$-Lyapunov function. □

Remark 1. In fact, Proposition 1 also proves that 0 is exponentially stable. Therefore, it can succeed only if the fixed point is actually exponentially stable.

Remark 2. Slope matrices have been used similarly in the more general context of hyperbolic fixed points in Wilczak and Zgliczyski (2016) (see Lemma 2.13 in that paper). The scope of Proposition 1 is smaller since it applies only to exponentially stable fixed points, but its statement is stronger for this smaller class of fixed points.

Checking if $[Q]$ is SND is NP-hard (Rohn, 1994). However, easily implementable polynomial sufficient conditions are available, e.g., checking the Sylvester criterion using the interval LU-decomposition for computing the determinant enclosures. In the special case of quadratic Lyapunov functions, $P(u) = P = D^2v(0)$ is symmetric, therefore (4) becomes $[Q] = P[A] + [A]^TP$.

4. Certified ERDA

In this section, we consider an uncertain vector field $f(x, \theta)$, θ being an uncertain parameter, whose domain is $\Theta = \{\theta \in [\theta] : g(\theta) \leq 0\}$, where the box $[\theta]$ is bounded. The function g defines the feasible uncertainties. We suppose that for all $\theta \in \Theta$ the system has a fixed point $x_\theta \in [x_\theta]$, an uncertain fixed point box enclosure $[x_\theta]$ being given. Let $\mathcal{D}_\theta$ be the DA of x_θ for the system $\dot{x} = f(x, \theta)$. The robust domain of attraction (RDA) is then defined by

$$\mathcal{D}_\Theta = \bigcap_{\theta\in\Theta} \mathcal{D}_\theta. \tag{5}$$

In Chesi (2011a) and Wu et al. (2014) only fixed points that do not depend on the uncertainty are considered. Notice that in the case where the fixed point actually depends on the uncertainty, the RDA may be empty although each uncertain system has a nonempty DA.

For a fixed $\theta \in \Theta$, let $f_\theta(u) = f(x_\theta + u, \theta)$, whose fixed point is 0. Note that $\mathcal{D}_\theta - x_\theta = \{x - x_\theta : x \in \mathcal{D}_\theta\}$ is the DA of 0. Given a box $[u] \ni 0$, we denote by $[A_\theta]$ a $(f_\theta, [u], 0)$-slope matrix enclosure and $A_\theta(u)$ its corresponding slope matrix. Since translations affect neither derivatives nor slope matrices, e.g., $Df_\theta(u) = D_xf(x_\theta + u, \theta)$, one verifies that for any fixed value $\theta \in \Theta$ both $[A_\theta] = [D_xf]([x], \theta)$ with $[x] = [u] + x_\theta$, and its Hansen improvement, whose (i, j) entry is

$$[A_\theta]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]([x]_1, \ldots, [x]_j, (x_\theta)_{j+1}, \ldots, (x_\theta)_n, \theta), \tag{6}$$

are $(f_\theta, [u], 0)$-slope matrix enclosures. A slope matrix enclosure $[A_\Theta]$ valid for all values $\theta \in \Theta$ is then $[A_\Theta] = [D_xf]([x], [\theta])$, with $[x] = [x_\theta] + [u]$, or its Hansen improvement

$$[A_\Theta]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]([x]_1, \ldots, [x]_j, [x_\theta]_{j+1}, \ldots, [x_\theta]_n, [\theta]). \tag{7}$$

These slope matrices are valid for all $\theta \in [\theta]$, which is a superset of Θ. This can sensibly over-approximate the slope matrix in the case where Θ is actually smaller than $[\theta]$, this issue being addressed below by splitting $[\theta]$.

We also consider a given Lyapunov candidate function $v : \mathbb{R}^n \to \mathbb{R}$, together with a PD quadratic minorant denoted by

$$w(u) = u^TWu \leq v(u), \tag{8}$$

with $W \in \mathbb{R}^{n\times n}$ being symmetric positive definite (SPD). Such a minorant implies in particular that v is radially unbounded and therefore has bounded level sets. If v is quadratic PD then $w = v$. Finding such a quadratic minorant valid in a given neighborhood can be modeled as a semi-infinite problem, and solved rigorously, e.g., using (Djelassi & Mitsos, 2016).

We finally consider a $(\nabla v, [u], 0)$-slope matrix enclosure $[P]$, which can be $[P] = [D^2v]([u])$, or its Hansen improvement $[P]_{ij} = [\frac{\partial^2 v}{\partial u_i \partial u_j}]([u]_1, \ldots, [u]_j, 0, \ldots, 0)$. The corresponding $(\nabla v, [u], 0)$-slope matrix is denoted by $P(u)$.

Given an approximation $\hat{x} \in \mathbb{R}^n$ of the fixed point(s) x_θ, the aim of this section is to compute $c > 0$ such that

$$\mathcal{V}(\hat{x}, c) = \hat{x} + \{u \in \mathbb{R}^n : v(u) \leq c\} \subseteq \mathcal{D}_\Theta, \tag{9}$$

where the Minkowski addition just translates the centered level-set of $\hat{x}$. Section 4.1 provides a test to decide whether $\mathcal{V}(\hat{x}, c)$ is inside the RDA. Section 4.2 provides a nonlinear optimization problem that enlarges this initial ERDA maximally so that Lie derivative remains strictly negative inside the enlarged ERDA for all uncertainties. In the case where the fixed point is perfectly known and does not depend on the uncertainties, this enlarged ERDA is exactly the largest estimate of the robust domain of attraction (LERDA), or the Largest estimate of the domain of attraction (LEDA) when there is no uncertain parameter, as defined in Chesi (2011a). Note that points with null Lie derivative inside the DA, which require LaSalle principle to be analyzed, are not included in LERDA or LEDA.

4.1. Certifying an ERDA

The following proposition requires the computation of a box enclosure $[u]$ of a level set $\mathcal{V}(0, c)$. Since w is a minorant of v, we have $\mathcal{V}(0, c) \subseteq \mathcal{W}(0, c)$, and w being quadratic PD,

$$[u] = c^{\frac{1}{2}}\,[-1, 1]\,\mathrm{diag}^{\frac{1}{2}}\left(W^{-1}\right) \supseteq \mathcal{W}(0, c), \tag{10}$$

where $\mathrm{diag}^{\frac{1}{2}}$ is the square root of the diagonal elements (the interval hull of an ellipsoid is given, e.g., in Domes and Neumaier (2011)).

Furthermore, for absorbing the uncertainty of the fixed point inside a level set of the Lyapunov function, we will need a Lipschitz constant for this Lyapunov candidate function. It is given by the following lemma.

Lemma 1. *The Lyapunov candidate function v is L-Lipschitz continuous inside $[u]$ (for the Euclidean distance) with $L = \|[P]([u])[u]\|_2$.*

Proof. Consider arbitrary $u, u' \in [u]$. Then using the mean-value theorem for several variables, there exists $u'' \in [u]$ such that $v(u) - v(u') = \nabla v(u'')^T(u - u')$. The proof is concluded noting that $|\nabla v(u'')^T(u - u')| \leq \|\nabla v(u'')\|_2\, \|u - u'\|_2$ by the Cauchy–Schwartz inequality, and that $\nabla v(u'') = P(u'')u'' \in [P]([u])[u]$. □

Proposition 2. *Given a Lyapunov candidate function v, $\hat{x} \in \mathbb{R}^n$, $c > 0$ and $[u] \supseteq \mathcal{V}(0, c)$, suppose that the interval matrix*

$$[Q] = [P]^T[A_\Theta] + [A_\Theta]^T[P] \tag{11}$$

is SND and that $c > \Delta := L\,\|[x_\theta] - \hat{x}\|_2$ with $L = \|[P]([u])[u]\|_2$. Then $\mathcal{V}(\hat{x}, c - \Delta) \subseteq \mathcal{D}_\Theta$.

Proof. We first prove that

$$\forall x_\theta \in [x_\theta],\ \mathcal{V}(\hat{x} - x_\theta, c - \Delta) \subseteq \mathcal{V}(0, c). \tag{12}$$

For arbitrary $\theta \in \Theta$ and $u' \in \mathcal{V}(\hat{x} - x_\theta, c - \Delta)$, we have $u' = \hat{x} - x_\theta + u$ with $v(u) \leq c - \Delta$. Therefore, $|v(u) - v(u')| \leq L\|u - u'\|_2 \leq \Delta$, which proves that $v(u') \leq v(u) + \Delta \leq c$, i.e., $u' \in \mathcal{V}(0, c)$, from which (12) follows.

For an arbitrary $\theta \in \Theta$, (11) and Proposition 1 prove that v is a $[u]$-Lyapunov function for f_θ. Furthermore, since $\mathcal{V}(0, c) \subseteq [u]$ by hypothesis, $\mathcal{V}(0, c) \subseteq \mathcal{D}_\theta - x_\theta$. As a consequence of (12), $\mathcal{V}(\hat{x} - x_\theta, c - \Delta) \subseteq \mathcal{D}_\Theta - x_\theta$ or equivalently $\mathcal{V}(\hat{x}, c - \Delta) \subseteq \mathcal{D}_\Theta$. □

Given $\hat{x}$ (usually chosen as $\hat{x} \approx \mathrm{mid}[x_\theta]$) and c, Proposition 2 can fail for one of the following two reasons: First, the interval matrix (11) is not SND. This can be the consequence of a too large $[u]$, which can be reduced by decreasing c, or a too large Θ, which can be reduced by splitting $[\theta]$ and applying Proposition 2 to all resulting subdomains $\Theta_i = \{\theta \in [\theta_i] : g(\theta) \le 0\}$ (empty subdomains being discarded). Then, the smallest ERDA obtained applying Proposition 2 to all subdomains Θ_i, i.e., $c - \Delta = \min_i c_i - \Delta_i$, gives rise to the ERDA $\mathcal{V}(\hat{x}, c - \Delta)$ valid for all uncertainties inside Θ. Second, the condition $c > \Delta$ is not satisfied. This does not happen if $[x_\theta]$ is small enough, in particular if the fixed point does not depend on the uncertainty. A too large uncertainty on the fixed point may prevent Proposition 2 from building any ERDA.

In the case where the fixed point does not depend on the uncertainty, i.e., $x_\theta = x_0$ for all $\theta \in \Theta$, and the vector field is smooth in a neighborhood of the fixed point Proposition 2 will succeed in building an ERDA provided that the Lyapunov candidate function is chosen correctly: Indeed, since (10) converges to zero as c does and the interval extension of the derivatives are convergent, decreasing c and splitting $[\theta]$ makes the slope matrix enclosures converge to thin matrices, and therefore Proposition 2 will eventually succeed provided that the real matrix $D^2v(0)Df(x_0) + Df(x_0)^T D^2v(0)$ is SND. This is the case when $D^2v(0)$ is chosen as the solution of the Lyapunov equation for the linearized system. Note that the worst-case complexity of splitting $[\theta]$ to obtain an ERDA is exponential with respect to the number of uncertain parameters.

However, if the vector field in non-smooth at the fixed point then generalized derivatives may not converge to thin intervals and the approach may fail to compute any ERDA.

4.2. Extending an ERDA

The following proposition shows how to increase a given ERDA by solving a nonlinear optimization problem.

Proposition 3. *Suppose that the Lyapunov function v is radially unbounded, which is entailed by $w(u) \le v(u)$, and consider a given ERDA $\mathcal{V}(\hat{x}, \underline{c})$. Let c^* be the (global) solution of the following nonlinear optimization problem:*

$$\min_{\substack{x\in\mathbb{R}^n,\ \theta\in\Theta,\ \underline{c}\le v(x-\hat{x}),\\ \nabla v(x-\hat{x})^T f(x,\theta)\ge 0}} v(x-\hat{x}). \tag{13}$$

Then for all positive lower bound $c^+ < c^*$, we have $\mathcal{V}(\hat{x}, c^+) \subseteq \mathcal{D}_\Theta$.

Proof. Since c^* is the global minimum, there is no feasible (x, θ) such that $v(x - \hat{x}) \le c^+ < c^*$, i.e., $\forall (x, \theta) \in \mathbb{R}^n \times \Theta$, $\underline{c} \le v(x - \hat{x}) \le c^+$ implies $\nabla v(x - \hat{x})^T f(x, \theta) < 0$. That is, for any $\theta \in \Theta$ and any trajectory $x(t)$ starting in $\mathcal{V}(\hat{x}, c^+) \setminus \mathcal{V}(\hat{x}, \underline{c})$, we have $\frac{d}{dt} v(x(t) - \hat{x}) \le \epsilon$ with

$$\epsilon = \max_{\substack{\underline{c}\le v(x-\hat{x})\le c^+\\ \theta\in\Theta}} \nabla v(x-\hat{x})^T f(x,\theta). \tag{14}$$

Since v is radially unbounded, the feasible set of this optimization problem is compact and $\epsilon < 0$. This trajectory therefore reaches $\mathcal{V}(\hat{x}, \underline{c})$ in a finite time, and then converges to x_θ because $\mathcal{V}(\hat{x}, \underline{c}) \subseteq \mathcal{D}_\Theta$. □

The initial ERDA $\mathcal{V}(\hat{x}, \underline{c})$ can be computed with Proposition 2. The nonlinear optimization problem (13) can be solved by branch and bound algorithms, which compute a certified lower bound $c^+ < c^*$ arbitrarily close to c^*.

It is crucial to note that the resolution has to be performed inside $\mathbb{R}^n \times \Theta$. Generally, branch and bound algorithms are not able to handle such unbounded domains. Some specific techniques were recently proposed in Domes and Neumaier (2016) to handle unbounded domains using quadratic minorants. In the same line, the quadratic PD minorant $w(u)$ gives rise to a specific contractor based on (10), which allows handling unbounded domains.

5. Experiments

In all experiments, an approximate fixed point $\hat{x}$ was provided and used as the center of the EDA. An interval Newton operator was used in order to compute a certified enclosure of the fixed point(s) starting at $\hat{x}$, including in Section 5.4 where the fixed point actually depends on the uncertainties. The Sylvester criterion was used to test whether an interval matrix is SPD or SND, the determinant being computed using an interval LU-decomposition.

A dichotomic search is used for finding the greatest value of c that can be validated using Proposition 2. The nonlinear problem (13) is solved using Ibex 2.6 (Araya, Trombettoni, Neveu, & Chabert, 2014; Chabert, 2011; Chabert & Jaulin, 2009) with standard settings with an initial upper bound for the objective $v(x - \hat{x})$ of 1000, which does not impact the rigorousness since the domain of the variables are kept unbounded, but sensibly speeds up the solving process.[1] Experiments are performed on a standard laptop equipped with an Intel i7 3 GHz and 8 Gb of memory. The C++ code used to perform these experiments is available at http://ibex-lib.org/papers.

Finally, we have extended the interval Hessian test (Delanoue et al., 2015) to tackle uncertain systems and non-quadratic Lyapunov functions, and included it in our LERDA computation framework. This allows a comprehensive comparison with our approach. A detailed comparison with Chesi (2011a) is also presented.

5.1. Example 1 from Chesi (2009)

This system is a non-polynomial system defined by $\dot{x}_1 = -x_1 + x_2 + \frac{1}{2}(\exp(x_1) - 1)$ and $x_2 = -x_1 - x_2 + x_1x_2 + x_1\cos(x_1)$. The origin is an attractive fixed point, which was studied using the quadratic Lyapunov function $v(x) = x_1^2 + x_2^2$ in Chesi (2009) and Wu et al. (2014). Using sixth order Taylor expansions of the non-polynomial operations, the LEDA $c = 0.3210$ is found in 4.8 s in Chesi (2009) (Matlab implementation with a standard personal computer), which is improved to $c = 0.3216$ in Wu et al. (2014).

For this system, the dichotomic search with a relative precision 10^{-4} using either the derivative enclosure of the slope matrix, its Hansen improvement or the interval Hessian test, computes the EDA $\mathcal{V}(0, c)$ with respectively $c^{\text{DER}} = 0.0248$, $c^{\text{HAN}} = 0.0565$ and $c^{\text{HES}} = 0.0098$. Computation timings are all less than 0.01 s. Solving the optimization problem (13) with a relative precision of 10^{-4} proves the LEDA $\mathcal{V}(0, c^*)$ satisfies $c^* \in [0.3210, 0.3212]$. The computation time is 0.1 s. The LEDA found in Chesi (2011a) is compatible with the enclosure found here. However, this enclosure is not compatible with the LEDA found in Wu et al. (2014). The following feasible point has been found by Ibex: $x_f = (0.4606476017903159, 0.3300125879647273)$ with $v(x_f) \le 0.32111$. This point has a positive Lie derivative and therefore contradicts (Wu et al., 2014).

5.2. Whirling pendulum (Chesi, 2009; Wu et al., 2014)

This model is a non-polynomial system defined by $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{c}{m}x_2 + \omega^2\sin(x_1)\cos(x_1) - \frac{g}{l}\sin(x_1)$, with $c = \frac{2}{10}$, $m = 1$, $\omega = \frac{9}{10}$ and $g = l = 10$. The origin is an attractive fixed point, which is used as the center of the computed EDA. The following quadratic Lyapunov function is used in Chesi (2009) and Wu et al. (2014): $v(x) = x_1^2 + x_1x_2 + 4x_2^2$. Using seventh order Taylor expansions of the non-polynomial operations, the LEDA $c = 0.6990$ was found in 8.5 s in Chesi (2011a) (Matlab

[1] This also has the advantage of letting the algorithm halt in case the feasible set is empty, e.g., when the fixed point is globally attractive. In this case, $\mathcal{V}(\hat{x}, 10^3)$ is proved to be inside the domain of attraction.

implementation with a standard personal computer), which was improved to $c = 0.6998$ in Wu et al. (2014).

The results obtained using the proposed approach with relative precision 10^{-4} are in line with the previous example: $c^{\text{DER}} = 0.0288$, $c^{\text{HAN}} = 0.0288$ and $c^{\text{HES}} = 0.0099$ are computed in less than 0.01 s. Then the enclosure $c^* \in [0.6992, 0.6993]$ is computed in 0.33 s. This LEDA enclosure is compatible with the one given in Chesi (2009). However, it is again not compatible with one given in Wu et al. (2014). The following feasible point has been found by IBEX: $x_f = (-0.7385018744233132, 0.3091449104899566)$ with $v(x_f) \le 0.6994$. This point has a positive Lie derivative and therefore contradicts (Wu et al., 2014).

5.3. Example 5.6 from Chesi (2011a)

This system is an uncertain polynomial system defined by $\dot{x}_1 = -2x_1 - x_2 - (1-\theta_2)x_1^2 + \theta_1 x_1 x_2$ and $\dot{x}_2 = -2x_1 - 3x_2 - \theta_2 x_1 x_2 + (2+\theta_1)x_2^2$. The domain for the uncertain parameters is $\Theta = \{\theta \in \mathbb{R}^2 : \theta_1^2 + \theta_2^2 \le 1\}$. The origin is an attractive fixed point, which does not depend on the uncertainties and is used as a center for the computed ERDA. The following quadratic Lyapunov function is used in Chesi (2011a): $v(x) = 3x_1^2 + x_1x_2 + x_2^2$. The LERDA $c = 0.303$ was found in Chesi (2011a) (no computation time is given).

The results obtained using the proposed approach with relative precision 10^{-3} are again in line with the previous examples: $c^{\text{DER}} = 0.0085$, $c^{\text{HAN}} = 0.0120$ and $c^{\text{HES}} = 0.0038$ are computed in less than 0.01 s. Then the enclosure $c^* \in [0.302, 0.304]$ is computed in 2.5 s. This LERDA enclosure is compatible with the one given in Chesi (2009).

5.4. A simple microbial growth process

This system is an uncertain non-polynomial system: $\dot{x} = x\left(\frac{s\,\mu_m}{k_s+s} - d\,\alpha\right)$ and $\dot{s} = d(s_i - s) - \frac{\mu_m\,k\,s\,x}{k_s+s}$, where $\alpha = 0.5$, $k = 10.53$, $d = 0.36$, $s_i = 5.7$, $\mu_m \in [1.19, 1.21]$ and $k_s \in [7.09, 7.11]$. There are therefore two uncertain parameters. This system was studied in Lin and Stadtherr (2006) where an initial condition was rigorously simulated and seemed to converge to a fixed point close to $\hat{x} = (0.845, 1.253)$. This fixed point actually depends on the uncertainties, therefore the ERDA computed here actually proves the convergence of trajectories towards an uncertain fixed point.

We use $\hat{x}$ as the center for the ERDA, and the matrix $P = ((10.398\ 0.2638); (0.2638\ 0.3645))$ obtained by solving the Lyapunov equation for the nominal $\hat{x}$-linearized system. Proposition 2 computes $c^{\text{HAN}} = 0.0967$ using the Hansen improvement in less than 0.01 s, but fails using the derivative-based slope matrix enclosure as well as using the interval Hessian test because the uncertain domain is too large. This initial ERDA is extended to $c^* \in [4.53, 4.54]$ in less than 0.1 s.

Finally, the uncertain domain Θ is split into 25 regular subintervals, leading to the success of the derivative-based slope enclosure with $c^{\text{DER}} = 0.0001$, but the interval Hessian test still fails. In fact, Delanoue et al. (2015) fails computing any EAD for this system: For arbitrarily small uncertainty subdomains we have $\Delta > c$ for some of them, which prevents any application of Proposition 2.

5.5. A linear system with saturation (Hu et al., 2005)

We consider the following linear system with saturated input, investigated in Hu et al. (2005): $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_1 + 5\max\{-1, \min\{1, -2x_1 - x_2\}\}$. Linearizing the system at the stable fixed point 0 and solving the corresponding Lyapunov equation, we select the Lyapunov candidate function $v_2(u) = 1.28x_1^2 + 0.11x_1x_2 + 0.11x_2^2$. In order to extend the EDA computed with this Lyapunov function, we also consider the degree 4 Lyapunov candidate function $v_4(x) = v_2(x) + \frac{1}{15}(x_1 + x_2)^4$.

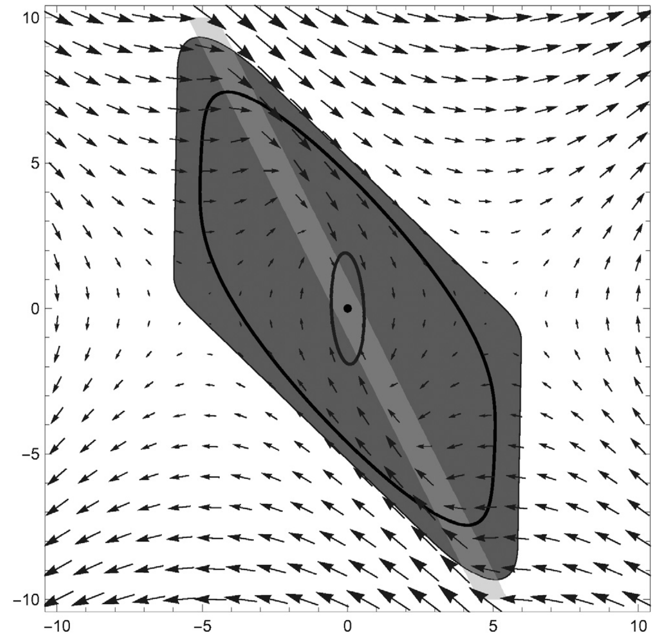

Fig. 1. The vector field of the saturated linear system of Section 5.5. In light-gray, the non-saturated area. The following EDA are displayed: In gray the one from Hu et al. (2005), which is the convex hull of two ellipsoids; the two black curves are boundaries of LEDA computed respectively using the quadratic and degree four Lyapunov functions.

The results obtained are summarized in Fig. 1: The LEDA obtained using the quadratic Lyapunov function is $c^* \in [0.400, 0.401]$, computed in 0.05 s. It is smaller than the one obtained with the degree 4 Lyapunov function, $c^* \in [32.54, 32.58]$, computed in 0.2 s. The latter is smaller, but still competitive with the one computed in Hu et al. (2005) also shown in Fig. 1. Note that the dichotomy search for the quadratic Lyapunov function using the derivative, its Hansen improvement or the interval Hessian test all give rise to $c = 0.043$, which is exactly where the interval hull of the computed ellipsoid reaches the saturation. On the other hand, using the slope arithmetic implemented in INTLAB (Rump, 1999) gives rise to $c = 0.1024$, which includes saturated inputs.

5.6. A scalable academic system

The scalable polynomial system $\dot{x} = -(1 - \prod x_i)x$ has a stable fixed point at the origin. We consider the Lyapunov function $v(x) = x^T x$, whose LEDA can be proved to be $\mathcal{V}(0, n)$.

The EDA found by applying Proposition 2 cannot be better than $\mathcal{V}(0, 1)$: Indeed, the slope matrix enclosure for this EDA is computed over its interval hull $[-1, 1]^n$, which contains the fixed point $(1, \ldots, 1)$. The values of c actually computed by dichotomy using Proposition 2 using the different slope matrix enclosures are shown in Fig. 2. We see that they are smaller than 1, but converge to 1 as the dimension increases. Timings are shown in Fig. 3. The resolution of the NLP (13) was run with a relative precision of 10^{-3} and a timeout of 15 min. Both the system and the Lyapunov function enjoy the same symmetry: All variables can be exchanged. Following Goldsztejn, Jermann, de Angulo, and Torras (2015), we break this symmetry by adding the symmetry breaking constraints (SBCs) $x_1 \le x_i$ for all $i \in \{2, \ldots, n\}$. The resolution of the NLP (13) allows computing a sharp enclosure of the LEDA without reaching the timeout until $n = 5$ ($n = 6$ using SBCs). For dimensions between $n = 6$ and $n = 11$ ($n = 7$ and $n = 15$ using SBCs),

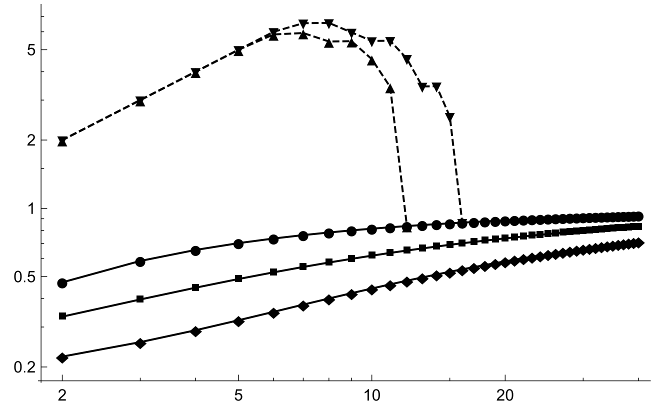

Fig. 2. Scalable academic problem: Radii for the different methods. Full line with diamond, square and disk: Dichotomy with Proposition 2 using respectively (Delanoue et al., 2015), derivative enclosure and Hansen improvement. Dashed line with upward and downward triangles: Solution of (13) resp. without and with symmetry breaking (timeout 15 min).

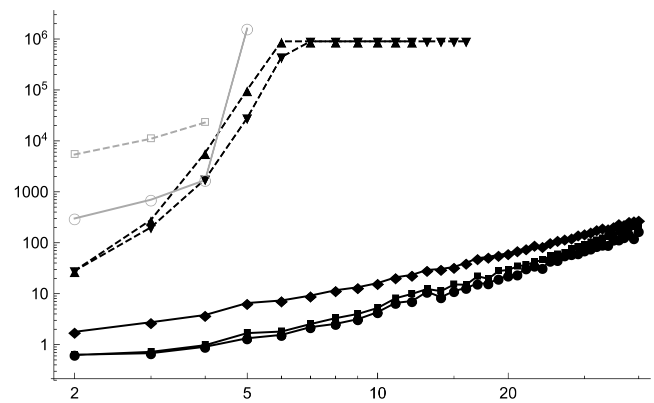

Fig. 3. Scalable academic problem: Timings for the different methods. Black: Same as Fig. 2. Gray, full line circle and dashed line empty square: SMRSOFT for resp. checking the EDA $\mathcal{V}(0, 0.5)$ and computing the LEDA.

the branch and bound still allows enlarging the EDA computed using Proposition 3 but does not provide a sharp enclosure of the LEDA anymore. No improvement over the initial EDA is provided for higher dimensions.

We use SMRSOFT (Chesi, 2011b) to compute EDA for this system. It is used to check the EDA $\mathcal{V}(0, 0.5)$, and to compute the LEDA of this system. Timings, shown in Fig. 3, are successfully computed up to $n = 5$ for the EDA and $n = 4$ for the LEDA, higher dimensions requiring too long computation and making our 2015*b* version of MATLAB crash.

6. Conclusion

The results presented above show the interest of the proposed approach: Given a Lyapunov candidate function, it successfully builds the LERDA on academic state-of-the-art examples tackled by the SOS approach. On the one hand, it has interesting advantages with respect to the SOS approach: It handles directly non-polynomials systems, while the SOS approach requires Taylor expansions with bounded remainders, as well as non-smooth systems and uncertainty dependent fixed points. Also, its application to the scalable benchmark has show its superiority in terms of computational effort, although the branch and bound algorithm also remains restricted to small dimensions. On the other hand, the SOS-based approach has a wider scope: Some other decision variables can be included inside the resulting LMI problem (e.g. controller parameters or coefficients of the Lyapunov function), and in some cases, it can prove the asymptotic stability of non exponentially stable fixed point using non quadratic functions (see, e.g., Example 1 in Papachristodoulou and Prajna (2002)).

Finally, the interval Lyapunov equation test proposed here presents several improvements with respect to the interval Hessian test (Delanoue et al., 2015): It requires only first-order derivatives interval evaluation, hence applicable to non-smooth systems, and actually computes larger ERDA.

Appendix. Slope matrix enclosure using intlab slope arithmetic

```
Min=@(a,b)0.5*(-abs(a-b)+a+b);
Max=@(a,b)0.5*(abs(a-b)+a+b);
f=@(x)[x(2);x(1) + 5*Max(-1, Min(1, -2*x(1) -
      x(2)))];
P=[1.28 0.055 ; 0.055 0.11];
c=0.32^2;
xi=midrad([0;0],sup(sqrt(intval(c))*sqrt(diag
     (inv(intval(P))))));
xs=slopeinit(mid(xi),xi);
fs=f(xs);
J=fs.s;
Q=P*J+transpose(J)*P;
mQ=-Q;
display(mQ(1,1));
display(mQ(1,1)*mQ(2,2)-mQ(1,2)*mQ(2,1));
```

The two displayed determinants 0.99 and $[0.0028, 0.9799]$ are positive, therefore Sylvester criterion proves that the interval matrix $-Q$ is definite positive. The test fails for $c \geq 0.33^2$.

References

Araya, Ignacio, Trombettoni, Gilles, Neveu, Bertrand, & Chabert, Gilles (2014). Upper bounding in inner regions for global optimization under inequality constraints. *Journal of Global Optimization*, *60*(2), 145–164.

Chabert, Gilles (2011). Ibex: a C++ library for constraint processing over real numbers. http://www.ibex-lib.org/.

Chabert, Gilles, & Jaulin, Luc (2009). Contractor Programming. *Artificial Intelligence*, *173*(11), 1079–1100.

Chesi, Graziano (2009). Estimating the domain of attraction for non-polynomial systems via LMI optimizations. *Automatica*, *45*(6), 1536–1541.

Chesi, Graziano (2011a). *Lecture notes in control and information sciences*: *Vol. 415*. *Domain of attraction: analysis and control via sos programming*. Springer.

Chesi, Graziano (2011b). SMRSOFT: a matlab toolbox for optimization over polynomials and dynamical systems study via SOS programming. http://www.eee.hku.hk/~chesi.

Clarke, Frank H. (1990). *Optimization and nonsmooth analysis*. Philadephia, PA: Society for Industrial and Applied Mathematics.

Delanoue, Nicolas, Jaulin, Luc, & Cottenceau, Bertrand (2015). An algorithm for computing a neighborhood included in the attraction domain of an asymptotically stable point. *Communications in Nonlinear Science and Numerical Simulation*, *21*(1), 181–189.

Djelassi, Hatim, & Mitsos, Alexander (2016). A hybrid discretization algorithm with guaranteed feasibility for the global solution of semi-infinite programs. *Journal of Global Optimization*, 1–27.

Domes, Ferenc, & Neumaier, Arnold (2011). Rigorous enclosures of ellipsoids and directed cholesky factorizations. *SIAM Journal on Matrix Analysis and Applications*, *32*(1), 262–285.

Domes, Ferenc, & Neumaier, Arnold (2016). Linear and parabolic relaxations for quadratic constraints. *Journal of Global Optimization*, *65*(3), 457–486.

Giesl, Peter A., & Hafstein, Sigurdur F. (2014). Revised CPA method to compute Lyapunov functions for nonlinear systems. *Journal of Mathematical Analysis and Applications*, *410*(1), 292–306.

Goldsztejn, Alexandre, Jermann, Christophe, de Angulo, Vincente Ruiz, & Torras, Carme (2015). Variable symmetry breaking in numeric constraint problems. *Artificial Intelligence*, *229*, 105–125.

Hansen, Eldon R. (1969). *Topics in interval analysis*. Oxford University Press.

Hu, Tingshu, Goebel, Rafal, Teel, Andrew R., & Lin, Zongli (2005). Conjugate Lyapunov functions for saturated linear systems. *Automatica*, *41*(11), 1949–1956.

Jansson, Christian, Chaykin, Denis, & Keil, Christian (2008). Rigorous error bounds for the optimal value in semidefinite programming. *SIAM Journal on Numerical Analysis*, *46*(1), 180–200.
Jaulin, Luc, Kieffer, Michel, Didrit, Olivier, & Walter, Eric (2001). *Applied interval analysis*. Springer.
Lin, Youdong, & Stadtherr, Mark A. (2006). Validated solution of ODEs with parametric uncertainties. In W. Marquardt, & C. Pantelides (Eds.), *Computer Aided Chemical Engineering*: *Vol. 21*. *ESCAPE 16 and PSE 9* (pp. 167–172). Elsevier.
Luk, Chuen Kit, & Chesi, Graziano (2015). Guaranteed estimates of the domain of attraction for a class of hybrid systems. *International Journal of Robust and Nonlinear Control*, *25*(17), 3270–3285.
Neumaier, Arnold (1990). *Interval methods for systems of equations*. Cambridge university press.
Papachristodoulou, Antonis, & Prajna, Stephen (2002). On the construction of Lyapunov functions using the sum of squares decomposition. In *Proceedings of the 41st IEEE conference on decision and control, 2002, Vol. 3* (pp. 3482–3487). IEEE.
Papachristodoulou, Antonis, & Prajna, Stephen (2005). Analysis of non-polynomial systems using the sum of squares decomposition. In Didier Henrion, & Andrea Garulli (Eds.), *Positive polynomials in control* (pp. 23–43). Springer Berlin Heidelberg.
Ratschan, Stefan, & She, Zhikun (2010). Providing a basin of attraction to a target region of polynomial systems by computation of lyapunov-like functions. *SIAM Journal on Control and Optimization*, *48*(7), 4377–4394.
Rohn, Jiri (1994). Positive definiteness and stability of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, *15*(1), 175–184.
Rump, Siegfried M. (1996). Expansion and estimation of the range of nonlinear functions. *Mathematics of Computation*, *65*(216), 1503–1512.
Rump, Siegfried M. (1999). INTLAB - INTerval LABoratory. In Tibor Csendes (Ed.), *Developments in reliable computing* (pp. 77–104). Dordrecht: Kluwer Academic Publishers.
Tibken, Bernd (2000). Estimation of the domain of attraction for polynomial systems via LMIs. In *Proceedings of the 39th IEEE conference on decision and control, 2000, Vol. 4* (pp. 3860–3864).
Wilczak, Daniel, & Zgliczyski, Piotr (2016). Connecting orbits for a singular nonautonomous real Ginzburg–Landau type equation. *SIAM Journal on Applied Dynamical Systems*, *15*(1), 495–525.
Wu, Min, Yang, Zhengfeng, & Lin, Wang (2014). Domain-of-attraction estimation for uncertain non-polynomial systems. *Communications in Nonlinear Science and Numerical Simulation*, *19*(9), 3044–3052.

Alexandre Goldsztejn received his engineer degree from the Institut Supérieur d'Electronique et du Numérique, Lille, France, in 2001, and his Ph.D. in Computer Science from the University of Nice-Sophia Antipolis in 2005. He has spent one year as a postdoctoral fellow in the University of Central Arkansas and the University of California Irvine. His research interests include interval analysis and its application to constraint satisfaction, nonlinear global optimization, robotics and control.

Gilles Chabert obtained an engineer degree from the Polytech'Nice engineering school (France) in 2001 and his Ph.D. in Computer Science from the University of Nice-Sophia Antipolis in 2007. He is associate professor at IMT Atlantique in Nantes since 2009, after a postdoc at ENSTA Bretagne in Brest. His research focuses on interval methods and he his the main developer of the IBEX C++ library.